

LARGE- N THEORY FROM THE AXIOMATIC POINT OF VIEW

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Abstract

The state space and observables for the leading order of the large- N theory are constructed. The obtained model ("theory of infinite number of fields") is shown to obey Wightman-type axioms (including invariance under boost transformations) and to be nontrivial (there are scattering processes, bound states, unstable particles etc). The considered class of exactly solvable relativistic quantum models involves good examples of theories containing such difficulties as volume divergences associated with the Haag theorem, Stueckelberg divergences and infinite renormalization of the wave function.

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Large-N expansion is widely used in quantum field theory [1, 2, 3]. This approximation allows us to obtain non-perturbative results and investigate the behavior of the Green functions, the effective action, dynamical and spontaneous symmetry breaking.

The traditional approaches to the $1/N$ -expansion enable us to evaluate different quantities mentioned above. However, some problems of the large-N theories remain to be understood. What are states and observables in the theory of infinite number of fields? Can one determine such a theory as a large-N limit?

From the axiomatic field theory point of view [4, 5, 6], the relativistic quantum field theory is constructed if:

- (i) the Hilbert state space \mathcal{H} is specified;
- (ii) the operators $U_g : \mathcal{H} \rightarrow \mathcal{H}$ corresponding to the Poincare transformations g are specified; the group property $U_{g_1}U_{g_2} = U_{g_1g_2}$ is satisfied;
- (iii) the field operators are constructed.

The introduced objects should obey certain (Wightman-type) axioms.

It happens that the axiomatic formulation of the large-N QFT can be obtained within the third-quantized approach developed recently in [7]. It is interesting that the large-N limit of QFT may be viewed as a theory of a variable number of fields. This is analogous to the statistical physics: the system of a large but fixed number of particles can be considered as a set of quasiparticles which can be created and annihilated. Analogously, the large-N field system can be treated from the "quasifield" point of view: there is an amplitude that there are no fields, that there is one field, two fields etc. Thus, the large-N limit of QFT is *not* a field theory in the usual treatment since one cannot define usual field operators. However, the property of the relativistic invariance remains. Moreover, we will introduce the analog of notion of field which is very useful for constructing boost transformations.

The models of infinite number of fields constructed in this paper seem to be remarkable from the point of view of the constructive field theory [8, 9]. The old problem of QFT is to construct the nontrivial model of field theory obeying all Wightman axioms. Such examples were constructed for the cases of 2- and 3-dimensional space-time. The models presented here are considered in higher dimensions.

The models considered in this paper are good examples of theories that contain such difficulties as Stueckelberg divergences [10] and volume divergences associated with the Haag theorem. There was a hypothesis [8] that the models with the Stueckelberg divergences cannot be constructed with the help of the Hamiltonian methods. However, we show this hypothesis to be not correct.

This paper is organized as follows.

As an example, we consider the $\lambda(\varphi^a\varphi^a)^2$ model in $(d+1)$ -dimensional space-time with the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a - \frac{m^2}{2}\varphi^a\varphi^a - \frac{\lambda}{4N}(\varphi^a\varphi^a)^2,$$

we sum over repeated indices $a = 1, \dots, N$, $\mu = 0, \dots, d$. In section II the $N = \infty$ -limit of the model is heuristically constructed. The Hamiltonian, momentum, angular momentum and boost generator are presented. It is heuristically shown that they formally obey usual commutation relations of the Poincare algebra. However, the divergences shows us that the obtained expressions are not mathematically well-defined.

Section III is devoted to the problem of renormalization of the Hamiltonian. The momentum and angular momentum are also investigated in section III. The spectral and vacuum axioms are checked.

It is not easy to construct operators of boost transformation (Lorentz rotation) and check the group properties. It is convenient first to introduce the composed field being an analog of the large-N operator $\sum_{a=1}^N \varphi^a(x_1)\dots\varphi^a(x_k)$. Such operators (multifields) being analogs of fields of ordinary QFT are constructed in section IV. They are shown to be operator distributions. The cyclic property of the vacuum state is checked. The invariance of multifields under spatial rotations and space-time translations is checked.

the representation of the Poincare group and check the relativistic invariance of the theory. The results of section IV are essentially used.

Section VI contains concluding remarks.

2 What is a theory of infinite number of fields?

This section deals with investigation of the theory of N fields, $\varphi^1, \dots, \varphi^N$ as $N \rightarrow \infty$. Such models were considered in context of calculations of physical quantities such as Green functions. It seems to be useful to formulate the $N = \infty$ -theory: to determine the state space, Poincare transformations, field operators etc.

2.1 Multifield canonical operator

In the functional Schrodinger representation states of the N -field system at time t are specified by the functionals $\Psi_N^t[\varphi^1(\cdot), \dots, \varphi^N(\cdot)]$ depending on the field configurations $\varphi^1(\mathbf{x}), \dots, \varphi^N(\mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_d)$. The inner product is formally written via the functional integral

$$(\Psi_N, \Psi_N) = \int D\varphi^1 \dots D\varphi^N |\Psi_N[\varphi^1(\cdot), \dots, \varphi^N(\cdot)]|^2.$$

The evolution equation has the form

$$i \frac{d}{dt} \Psi_N^t = \mathcal{H}_N \Psi_N^t \quad (1)$$

with the Hamiltonian presented as a sum of the free Hamiltonian and interaction

$$\mathcal{H}_N = \int d\mathbf{x} \left[-\frac{1}{2} \frac{\delta^2}{\delta \varphi^a(\mathbf{x}) \delta \varphi^a(\mathbf{x})} + \frac{1}{2} (\nabla \varphi^a)(\mathbf{x}) (\nabla \varphi^a)(\mathbf{x}) + \frac{m^2}{2} \varphi^a(\mathbf{x}) \varphi^a(\mathbf{x}) + \frac{\lambda}{4N} (\varphi^a(\mathbf{x}) \varphi^a(\mathbf{x}))^2 \right] \quad (2)$$

If one considers states of a few number of particles in comparison with N , one can suppose that almost all fields are in the vacuum state. This treatment leads us to the following structure of the wave functional Ψ_N^t . If all fields $\varphi^1, \dots, \varphi^N$ are in the same (vacuum) state, the N -field state Ψ_N is

$$\Psi_N[\varphi^1(\cdot), \dots, \varphi^N(\cdot)] = c \Phi_0[\varphi^1(\cdot)] \dots \Phi_0[\varphi^N(\cdot)]. \quad (3)$$

If $(N-1)$ fields are in the state Φ_0 , while 1 field is in the state f_1 , the N -field state can be written as

$$\Psi_N[\varphi^1(\cdot), \dots, \varphi^N(\cdot)] = \frac{1}{\sqrt{N}} \sum_{a=1}^N \Phi_0[\varphi^1(\cdot)] \dots \Phi_0[\varphi^{a-1}(\cdot)] f_1[\varphi^a(\cdot)] \Phi_0[\varphi^{a+1}(\cdot)] \dots \Phi_0[\varphi^N(\cdot)] \quad (4)$$

Without loss of generality, one can suppose that $(\Phi_0, f_1) = 0$. Otherwise, one could decompose the functional f_1 into two parts, one of them being proportional to Φ_0 , another being orthogonal to Φ_0 . The case $f_1 = \text{const} \Phi_0$ does not lead to a new functional since expressions (3) and (4) coincide then.

Analogously, the state corresponding to $(N-k)$ fields in the vacuum state and k fields in the state $f_k(\varphi^1, \dots, \varphi^k)$ being symmetric with respect to transpositions of $\varphi^1, \dots, \varphi^k$ and satisfying the orthogonality condition

$$\int D\varphi_1 \Phi_0^*[\varphi^1(\cdot)] f_k[\varphi^1(\cdot), \dots, \varphi^k(\cdot)] = 0 \quad (5)$$

has the form

$$\Psi_N[\varphi^1(\cdot), \dots, \varphi^N(\cdot)] = \frac{1}{\sqrt{N^k k!}} \sum_{1 \leq a_1 \neq \dots \neq a_k \leq N} f_k[\varphi^{a_1}(\cdot), \dots, \varphi^{a_k}(\cdot)] \prod_{a \neq a_1 \dots a_k} \Phi_0[\varphi^a(\cdot)]. \quad (6)$$

$\|f_k\|$. This is the most general form of a state "with a few number of particles", provided that one takes into account symmetric functionals Ψ only. Nonsymmetric functionals are investigated in appendix B.

We see that symmetric states in the theory of a large number of fields occur to be specified by infinite sets

$$f = \begin{pmatrix} f_0 \\ f_1[\varphi^1(\cdot)] \\ \dots \\ f_k[\varphi^1(\cdot), \dots, \varphi^k(\cdot)] \\ \dots \end{pmatrix} \quad (7)$$

where f_k are symmetric functionals satisfying eq.(5). One can say that f_k is a probability amplitude that k fields are in the non-vacuum state. We see that the theory of a large number of fields is equivalent to the theory of a variable number of fields. This observation is analogous to the quasiparticle conception in statistical physics.

The mapping $K_N : f \mapsto \Psi_N$ of the form

$$(K_N f)[\varphi^1(\cdot), \dots, \varphi^N(\cdot)] = \sum_{k=0}^N \frac{1}{\sqrt{N^k k!}} \sum_{1 \leq a_1 \neq \dots \neq a_k \leq N} f_k[\varphi^{a_1}(\cdot), \dots, \varphi^{a_k}(\cdot)] \prod_{a \neq a_1 \dots a_k} \Phi_0[\varphi^a(\cdot)] \quad (8)$$

will be called as a multifield canonical operator analogously to the multiparticle canonical operator used in statistical physics [7]. The orthogonality condition (5) implies that

$$\|K_N f\|^2 = \sum_{k=0}^N \frac{N!}{N^k (N-k)!} \|f_k\|^2 \xrightarrow{N \rightarrow \infty} \sum_{k=0}^{\infty} \|f_k\|^2. \quad (9)$$

We see that sets (7) may be identified with states of the system of $N = \infty$ fields, while the relation (9) can be considered as an argument that the norm of a state should be chosen as

$$\|f\|^2 = \sum_{k=0}^{\infty} \|f_k\|^2.$$

Thus, decomposition (8) gives us a relationship between the theory of a large number of fields and the theory of a variable number of fields.

2.2 Representation of operators

Let us write operators of physical quantities in the representation (7). It will be convenient to present them via the third-quantized creation and annihilation operators which can be introduced as follows [7]. The creation operator $A^+[\varphi(\cdot)]$ increases the number of fields, i.e. transforms the set $f = (0, 0, \dots, 0, f_{k-1}, 0, \dots)$ into $(0, \dots, 0, (A^+[\varphi(\cdot)]f)_k, 0, \dots)$, The functional $(A^+[\varphi(\cdot)]f)_k[\varphi^1(\cdot), \dots, \varphi^k(\cdot)]$ being the k -th component of the set $(A^+[\varphi(\cdot)]f)$ is expressed via the $(k-1)$ -th component of f :

$$(A^+[\varphi(\cdot)]f)_k[\varphi^1(\cdot), \dots, \varphi^k(\cdot)] = \frac{1}{\sqrt{k}} \sum_{a=1}^k \delta(\varphi(\cdot) - \varphi^a(\cdot)) f_{k-1}[\varphi^1(\cdot), \dots, \varphi^{a-1}(\cdot), \varphi^{a+1}(\cdot), \dots, \varphi^k(\cdot)]. \quad (10)$$

The annihilation operator $A^-[\varphi(\cdot)]$ is

$$(A^-[\varphi(\cdot)]f)_{k-1}[\varphi^1(\cdot), \dots, \varphi^{k-1}(\cdot)] = \sqrt{k} f_k[\varphi(\cdot), \varphi^1(\cdot), \dots, \varphi^{k-1}(\cdot)]. \quad (11)$$

annihilation operators:

$$\begin{aligned}\tilde{A}^+[\varphi(\cdot)] &= A^+[\varphi(\cdot)] - \Phi_0^*[\varphi(\cdot)] \int D\phi \Phi_0[\phi(\cdot)] A^+[\phi(\cdot)] \\ \tilde{A}^-[\varphi(\cdot)] &= A^-[\varphi(\cdot)] - \Phi_0[\varphi(\cdot)] \int D\phi \Phi_0^*[\phi(\cdot)] A^-[\phi(\cdot)]\end{aligned}$$

To write operators in the representation (7), consider the orthonormal basis (Φ_0, Φ_1, \dots) in the space of functionals,

$$\int D\varphi \Phi_i^*[\varphi(\cdot)] \Phi_j[\varphi(\cdot)] = \delta_{ij}$$

which contains the vacuum functional Φ_0 entering to expression (8). Investigate the following “elementary” operators

$$\mathcal{O}_N^{ij} \Psi[\varphi^1(\cdot), \dots, \varphi^N(\cdot)] = \sum_{a=1}^N \Phi_i[\varphi^a(\cdot)] \int D\phi \Phi_j^*[\phi(\cdot)] \Psi[\varphi^1(\cdot), \dots, \varphi^{a-1}(\cdot), \phi(\cdot), \varphi^{a+1}(\cdot), \dots, \varphi^N(\cdot)]. \quad (12)$$

Apply them to the expression (8). It is necessary to distinguish 4 cases.

(i) $i = j = 0$.

Due to the condition (5), one has

$$(\mathcal{O}_N^{00} K_N f)[\varphi^1(\cdot), \dots, \varphi^N(\cdot)] = \sum_{k=0}^N (N-k) \frac{1}{\sqrt{N^k k!}} \sum_{1 \leq a_1 \neq \dots \neq a_k \leq N} f_k[\varphi^{a_1}(\cdot), \dots, \varphi^{a_k}(\cdot)] \prod_{a \neq a_1 \dots a_k} \Phi_0[\varphi^a(\cdot)].$$

This means that the operator \mathcal{O}_N^{00} acts in the space (7) as $N - \hat{n}$, i.e.

$$\mathcal{O}_N^{00} K_N f = K_N (N - \hat{n}) f,$$

where $\hat{n} = \int D\varphi \tilde{A}^+[\varphi(\cdot)] \tilde{A}^-[\varphi(\cdot)]$ is the operator of number of fields,

$$(\hat{n} f)_k = k f_k.$$

(ii) $i = 0, j \neq 0$.

It follows from the symmetry condition that

$$\begin{aligned}(\mathcal{O}_N^{0j} K_N f)[\varphi^1(\cdot), \dots, \varphi^N(\cdot)] &= \sum_{k=0}^N \frac{1}{\sqrt{N^k k!}} \sum_{p=1}^k \frac{1}{\sqrt{k}} \sum_{1 \leq a_1 \neq \dots \neq a_k \leq N} \Phi_0[\varphi^{a_p}(\cdot)] \\ &\times \left(\int D\phi \tilde{A}^-[\phi(\cdot)] \Phi_j^*[\phi(\cdot)] f \right)_{k-1} [\varphi^{a_1}(\cdot), \dots, \varphi^{a_{p-1}}(\cdot), \varphi^{a_{p+1}}(\cdot), \dots, \varphi^{a_k}(\cdot)] \prod_{a \neq a_1 \dots a_k} \Phi_0[\varphi^a(\cdot)].\end{aligned} \quad (13)$$

After redefining $a_1 = b_1, \dots, a_{p-1} = b_{p-1}, a_{p+1} = b_p, \dots, a_k = b_{k-1}, a_k = b$ we obtain that the symbol \sum_k can be substituted by k , while \sum_j transforms to $(N - k + 1)$. Thus, one obtains the following commutation rule:

$$\mathcal{O}_N^{0j} K_N f = K_N (N - \hat{n}) \frac{1}{\sqrt{N}} \int D\phi \tilde{A}^-[\phi(\cdot)] \Phi_j^*[\phi(\cdot)] f.$$

(iii) $i \neq 0, j = 0$.

Due to eq.(5), we have

$$\begin{aligned}(\mathcal{O}_N^{i0} K_N f)[\varphi^1(\cdot), \dots, \varphi^N(\cdot)] &= \sum_{k=0}^N \frac{1}{\sqrt{N^k k!}} \sum_{1 \leq a_1 \neq \dots \neq a_k \leq N} f_k[\varphi^{i_1}(\cdot), \dots, \varphi^{i_k}(\cdot)] \\ &\times \sum_{a \neq a_1 \dots a_k} \Phi_i[\varphi^a(\cdot)] \prod_{b \neq a, a_1 \dots a_k} \Phi_0[\varphi^b(\cdot)].\end{aligned} \quad (14)$$

$$\mathcal{O}_N^{i0} K_N f = K_N \sqrt{N} \int D\phi \tilde{A}^+[\phi(\cdot)] \Phi_i[\phi(\cdot)] f.$$

(iv) $i \neq 0, j \neq 0$.

Analogously, we find that

$$\mathcal{O}_N^{ij} K_N f = K_N \int D\varphi \tilde{A}^+[\varphi(\cdot)] \Phi_i[\varphi(\cdot)] \int D\phi \tilde{A}^+[\phi(\cdot)] \Phi_j^*[\phi(\cdot)]$$

Any operator can be represented via elementary operators (12). Consider an example.

The operator $\sum_{a=1}^N \varphi^a(\mathbf{x}) \varphi^a(\mathbf{x})$ is expressed as

$$\sum_{a=1}^N \varphi^a(\mathbf{x}) \varphi^a(\mathbf{x}) = \sum_{ij=0}^{\infty} \int D\phi \Phi_i^*[\phi(\cdot)] \phi(\mathbf{x}) \phi(\mathbf{x}) \Phi_j[\phi(\cdot)] \mathcal{O}_N^{ij}.$$

Therefore, the following commutation rule takes place:

$$\lambda \sum_{a=1}^N \varphi^a(\mathbf{x}) \varphi^a(\mathbf{x}) K_N = K_N \tilde{\mathcal{Q}}_N(\mathbf{x}),$$

where the operator $\tilde{\mathcal{Q}}_N(\mathbf{x})$ consists of the constant term of order $O(N)$, the linear in creation-annihilation operators term of order $O(\sqrt{N})$ and the regular as $N \rightarrow \infty$ term which is quadratic in creation and annihilation operators:

$$\begin{aligned} \tilde{\mathcal{Q}}_N(\mathbf{x}) &= \lambda(N - \hat{n})(\Phi_0, \phi(\mathbf{x})\phi(\mathbf{x})\Phi_0) + \lambda\sqrt{N} \int D\phi \tilde{A}^+[\phi(\cdot)] \phi(\mathbf{x}) \phi(\mathbf{x}) \Phi_0[\phi(\cdot)] \\ &+ \frac{\lambda}{\sqrt{N}}(N - \hat{n}) \int D\phi \tilde{A}^-[\phi(\cdot)] \phi(\mathbf{x}) \phi(\mathbf{x}) \Phi_0^*[\phi(\cdot)] + \lambda \int D\phi \tilde{A}^+[\phi(\cdot)] \phi(\mathbf{x}) \phi(\mathbf{x}) \tilde{A}^-[\phi(\cdot)]. \end{aligned} \quad (15)$$

2.3 Evolution equation at $N = \infty$

Analogously to the previous subsection, the operators $\sum_{a=1}^n (\nabla \varphi^a)^2(\mathbf{x})$ and $\sum_{a=1}^N (-\frac{\delta^2}{\delta \varphi^a(\mathbf{x}) \delta \varphi^a(\mathbf{x})})$ can be also written in the representation (7). Since the Hamiltonian (2) contains the considered operator expressions only, it can be also commuted with the multifield canonical operator,

$$\mathcal{H}_N K_N = K_N \tilde{\mathcal{H}}_N.$$

The transformed Hamiltonian $\tilde{\mathcal{H}}_N$ is

$$\begin{aligned} \tilde{\mathcal{H}}_N &= \int d\mathbf{x} \left[(N - \hat{n})(\Phi_0, \mathcal{E}_0(\mathbf{x})\Phi_0) + \sqrt{N} \int D\phi \tilde{A}^+[\phi(\cdot)] \mathcal{E}_0(\mathbf{x}) \Phi_0[\phi(\cdot)] \right. \\ &\left. + \frac{1}{\sqrt{N}}(N - \hat{n}) \int D\phi \Phi_0^*[\phi(\cdot)] \mathcal{E}_0(\mathbf{x}) \tilde{A}^-[\phi(\cdot)] + \int D\phi \tilde{A}^+[\phi(\cdot)] \mathcal{E}_0(\mathbf{x}) \tilde{A}^-[\phi(\cdot)] + \frac{1}{4N\lambda} \tilde{\mathcal{Q}}_N^2(\mathbf{x}), \right] \end{aligned} \quad (16)$$

where

$$\mathcal{E}_0(\mathbf{x}) = -\frac{1}{2} \frac{\delta^2}{\delta \phi(\mathbf{x}) \delta \phi(\mathbf{x})} + \frac{1}{2} (\nabla \phi)^2(\mathbf{x}) + \frac{m^2}{2} \phi^2(\mathbf{x}).$$

Expression (16) contains the terms of order $O(N)$, $O(N^{1/2})$, $O(1)$ and the terms damping as $N \rightarrow \infty$:

$$\tilde{\mathcal{H}}_N = N \tilde{\mathcal{H}}^0 + N^{1/2} \tilde{\mathcal{H}}^1 + \tilde{\mathcal{H}}^2 + O(N^{-1/2}). \quad (17)$$

$$\tilde{\mathcal{H}}^0 = \int d\mathbf{x}(\Phi_0, \mathcal{E}_0(\mathbf{x})\Phi_0) + \frac{\lambda}{4} \int d\mathbf{x}(\Phi_0, \phi(\mathbf{x})\phi(\mathbf{x})\Phi_0)^2.$$

As usual in QFT, the vacuum energy is set to zero by adding a constant to the Hamiltonian, so that the Hamiltonian is defined up to a constant, and the quantity $\tilde{\mathcal{H}}^0$ can be neglected.

The operator $\tilde{\mathcal{H}}^1$ is a linear combination of creation and annihilation operators:

$$\tilde{\mathcal{H}}^1 = \int D\phi \tilde{A}^+[\phi(\cdot)]Z[\phi(\cdot)] + \int D\phi \tilde{A}^-[\phi(\cdot)]Z^*[\phi(\cdot)]$$

with

$$Z[\phi(\cdot)] = \int d\mathbf{x} \left[-\frac{1}{2} \frac{\delta^2}{\delta\phi(\mathbf{x})\delta\phi(\mathbf{x})} + \frac{1}{2} (\nabla\phi(\mathbf{x}))^2 + \frac{m^2 + \lambda(\Phi_0, \phi^2(\mathbf{x})\Phi_0)}{2} \phi^2(\mathbf{x}) \right] \Phi_0[\phi(\cdot)].$$

The operator $\tilde{\mathcal{H}}^1$ vanishes if and only if

$$Z = \text{const}\Phi_0. \quad (18)$$

We choose the functional Φ_0 to be a vacuum state functional for the field of the mass μ ,

$$\Phi_0[\phi(\cdot)] = \text{const} \exp\left[-\frac{1}{2} \int d\mathbf{x} \phi(\mathbf{x}) \sqrt{-\Delta + \mu^2} \phi(\mathbf{x})\right], \quad (19)$$

so that eq.(18) will take the form

$$\mu^2 = m^2 + \lambda(\Phi_0, \phi^2(\mathbf{x})\Phi_0). \quad (20)$$

This is a well-known equation in the $1/N$ -expansion theory (see, for example, [2]).

The remaining nonvanishing as $N \rightarrow \infty$ part of the Hamiltonian is quadratic in creation and annihilation operators,

$$\tilde{H} \equiv \tilde{\mathcal{H}}^2 = \int D\phi \tilde{A}^+[\phi(\cdot)] : \int d\mathbf{x} \mathcal{E}(\mathbf{x}) : \tilde{A}^-[\phi(\cdot)] + \frac{\lambda}{4} \int d\mathbf{x} Q_0^2(\mathbf{x}). \quad (21)$$

where $: \hat{O} := \hat{O} - (\Phi_0, \hat{O}\Phi_0)$,

$$\mathcal{E}(\mathbf{x}) = \left[-\frac{1}{2} \frac{\delta^2}{\delta\phi(\mathbf{x})\delta\phi(\mathbf{x})} + \frac{1}{2} (\nabla\phi(\mathbf{x}))^2 + \frac{\mu^2}{2} \phi^2(\mathbf{x}) \right]$$

$$Q_0(\mathbf{x}) = \int D\phi (\tilde{A}^+[\phi(\cdot)] + \tilde{A}^-[\phi(\cdot)]) \phi^2(\mathbf{x}) \Phi_0[\phi(\cdot)].$$

Since the term (21) is the only term remaining as $N = \infty$, one can say that the theory of $N = \infty$ fields is as follows. States in this theory are sets (7) obeying eq.(5). The Hamiltonian of the model has the form (21), the evolution equation is $if = \tilde{H}f$.

2.4 Representation of the Poincare algebra

We have specified the state space of the theory of $N = \infty$ fields and evolution operator. However, to construct the *relativistic* quantum theory, it is necessary to specify the operators $U_{\Lambda,a}$ corresponding to Poincare transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \quad \mu, \nu = \overline{0, d},$$

where the matrix Λ of Lorentz transformation satisfies the property

$$\Lambda^T g \Lambda = g$$

formations is

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2),$$

so that any Poincare transformation can be presented as $(\Lambda, a) = (0, a)(\Lambda, 0)$. Furthermore, one can introduce the local coordinates $\theta_{\lambda\mu}$ ($\lambda, \mu = \overline{1, d}$, $\theta_{\lambda\mu} = -\theta_{\mu\lambda}$) on the Lorentz group [6], such that

$$\Lambda = \exp\left(\frac{1}{2}\theta_{\lambda\mu}l^{\lambda\mu}\right)$$

with

$$(l^{\lambda\mu})_{\beta}^{\alpha} = -g^{\lambda\alpha}\delta_{\beta}^{\mu} + g^{\mu\alpha}\delta_{\beta}^{\lambda}.$$

The operators $U_{\Lambda, a}$ are required to form the representation of the Poincare group, so that

$$U_{\Lambda_1, a_1}U_{\Lambda_2, a_2} = U_{(\Lambda_1\Lambda_2), (a_1 + \Lambda_1 a_2)}$$

Making use of the theory of representations of the Lie groups, one finds [6]

$$U_{\Lambda, a} = \exp(i\tilde{P}^{\mu}a_{\mu})\exp\left(\frac{i}{2}\tilde{M}^{\lambda\mu}\theta_{\lambda\mu}\right)$$

for some operators \tilde{P}^{μ} and $\tilde{M}^{\lambda\mu}$ obeying the commutation relations of the Poincare algebra

$$\begin{aligned} [\tilde{P}^{\lambda}, \tilde{P}^{\mu}] &= 0, & [\tilde{M}^{\lambda\mu}, \tilde{P}^{\nu}] &= i(g^{\mu\nu}\tilde{P}^{\lambda} - g^{\lambda\nu}\tilde{P}^{\mu}), \\ [\tilde{M}^{\lambda\mu}, \tilde{M}^{\rho\sigma}] &= -i(g^{\lambda\rho}\tilde{M}^{\mu\sigma} - g^{\mu\rho}\tilde{M}^{\lambda\sigma} + g^{\mu\sigma}\tilde{M}^{\lambda\rho} - g^{\lambda\sigma}\tilde{M}^{\mu\rho}) \end{aligned} \quad (22)$$

Let us construct the operators \tilde{P}^{λ} and $\tilde{M}^{\lambda\mu}$ for the $N = \infty$ -theory. For the N -field theory, one has [13]:

$$\mathcal{P}_N^{\mu} = \int d\mathbf{x} \mathcal{T}_N^{\mu 0}(\mathbf{x}), \quad \mathcal{M}_N^{\mu\lambda} = \int d\mathbf{x} (x^{\mu} \mathcal{T}_N^{\lambda 0}(\mathbf{x}) - x^{\lambda} \mathcal{T}_N^{\mu 0}(\mathbf{x})),$$

where we integrate over surface $x^0 = 0$, while

$$\mathcal{T}_N^{00}(\mathbf{x}) = -\frac{1}{2} \frac{\delta^2}{\delta\varphi^a(\mathbf{x})\delta\varphi^a(\mathbf{x})} + \frac{1}{2} (\nabla\varphi^a)(\mathbf{x})(\nabla\varphi^a)(\mathbf{x}) + \frac{m^2}{2} \varphi^a(\mathbf{x})\varphi^a(\mathbf{x}) + \frac{\lambda}{4N} (\varphi^a(\mathbf{x})\varphi^a(\mathbf{x}))^2,$$

$$\mathcal{T}_N^{k0}(\mathbf{x}) = \sum_{a=1}^N (\partial^k \varphi^a(\mathbf{x}) \frac{1}{i} \frac{\delta}{\delta\varphi^a(\mathbf{x})})$$

Let us commute these operators with the multifield canonical operator,

$$\mathcal{P}_N^{\mu} K_N = K_N \tilde{\mathcal{P}}_N^{\mu}, \quad \mathcal{M}_N^{\mu\nu} K_N = K_N \tilde{\mathcal{M}}_N^{\mu\nu},$$

expand the result in $1/N$:

$$\tilde{\mathcal{P}}_N^{\mu} = N\tilde{\mathcal{P}}^{\mu,0} + N^{1/2}\tilde{\mathcal{P}}^{\mu,1} + \tilde{\mathcal{P}}^{\mu,2} + \dots, \quad \tilde{\mathcal{M}}_N^{\mu\nu} = N\tilde{\mathcal{M}}^{\mu\nu,0} + N^{1/2}\tilde{\mathcal{M}}^{\mu\nu,1} + \tilde{\mathcal{M}}^{\mu\nu,2} + \dots$$

It will be shown that the operators $\tilde{\mathcal{P}}^{\mu,0}$, $\tilde{\mathcal{P}}^{\mu,1}$, $\tilde{\mathcal{M}}^{\mu\nu,0}$, $\tilde{\mathcal{M}}^{\mu\nu,1}$ vanish, so that the remaining nonvanishing at $N = \infty$ parts

$$\tilde{\mathcal{P}}^{\mu} = \tilde{\mathcal{P}}^{\mu,2}, \quad \tilde{\mathcal{M}}^{\mu\nu} = \tilde{\mathcal{M}}^{\mu\nu,2}$$

should be viewed as generators of Poincare transformations in the $N = \infty$ -theory.

Remind also that the operator $\tilde{\mathcal{P}}_N^0 = \tilde{H}$ has been already constructed in the previous subsection.

Consider the operator

$$\mathcal{P}_N^k = \int d\mathbf{x} \sum_{a=1}^N \partial^k \varphi^a(\mathbf{x}) \frac{1}{i} \frac{\delta}{\delta\varphi^a(\mathbf{x})}.$$

$$\begin{aligned}\tilde{\mathcal{P}}^{k,0} &= (\Phi_0, \int d\mathbf{x} \partial^k \phi(\mathbf{x}) \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})} \Phi_0), \\ \tilde{\mathcal{P}}^{k,1} &= \int D\phi (\tilde{A}^+[\phi(\cdot)] Z^k[\phi(\cdot)] + \tilde{A}^-[\phi(\cdot)] Z^{k*}[\phi(\cdot)]),\end{aligned}\tag{23}$$

where

$$Z^k[\phi(\cdot)] = \int d\mathbf{x} \partial^k \phi(\mathbf{x}) \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})} \Phi_0[\phi(\cdot)].$$

Since Φ_0 has been chosen to be a vacuum functional for the field of the mass μ , while the operator $\int d\mathbf{x} \partial^k \phi(\mathbf{x}) \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})}$ is a momentum operator for the functional Schrodinger representation, one has $Z^k = 0$, $\tilde{\mathcal{P}}^{k,0} = 0$, $\tilde{\mathcal{P}}^{k,1} = 0$. Thus, the operator

$$\tilde{P}^k \equiv \tilde{\mathcal{P}}^{k,2} = \int D\phi \tilde{A}^+[\phi(\cdot)] : \int d\mathbf{x} \partial^k \phi(\mathbf{x}) \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})} : \tilde{A}^+[\phi(\cdot)]\tag{24}$$

can be viewed as a momentum operator in the $N = \infty$ theory.

Analogously, we find that

$$\tilde{M}^{ml} = \tilde{\mathcal{M}}^{ml,2} = \int D\phi \tilde{A}^+[\phi(\cdot)] : \int d\mathbf{x} (x^m \partial^l \phi(\mathbf{x}) - x^l \partial^m \phi(\mathbf{x})) \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})} : \tilde{A}^-[\phi(\cdot)].\tag{25}$$

The boost operator presented as

$$\mathcal{M}^{k0} = \int d\mathbf{x} x^k \mathcal{T}^{00}(\mathbf{x})$$

after commuting with the multiparticle canonical operator gives us:

$$\tilde{\mathcal{M}}^{k0,0} = \int d\mathbf{x} x^k (\Phi^0, \mathcal{E}_0(\mathbf{x}) \Phi_0) + \frac{\lambda}{4} \int d\mathbf{x} x^k (\Phi_0, \phi(\mathbf{x}) \phi(\mathbf{x}) \Phi_0).$$

Since the integrand is an odd function with respect to x^m , it seems to be natural that $\tilde{\mathcal{M}}^{m0,0} = 0$. The operator $\tilde{\mathcal{M}}^{k0,1}$ has the structure (23) with

$$Z^k[\phi(\cdot)] = \int d\mathbf{x} x^k \mathcal{E}(\mathbf{x}) \Phi_0[\phi(\cdot)].$$

Since the vacuum state Φ_0 is invariant under boost transformations, while

$$\int d\mathbf{x} x^k \mathcal{E}(\mathbf{x})$$

is a boost generator, one has $Z^k = 0$ and $\tilde{\mathcal{M}}^{k0,1} = 0$. The remaining term is

$$\tilde{M}^{k0} = \tilde{\mathcal{M}}^{k0,2} = \int D\phi \tilde{A}^+[\phi(\cdot)] \int d\mathbf{x} x^k : \mathcal{E}(\mathbf{x}) : \tilde{A}^-[\phi(\cdot)] + \frac{\lambda}{4} \int d\mathbf{x} x^k Q_0^2(\mathbf{x}).\tag{26}$$

The commutation relations (22) are formally satisfied. Namely, the operators (21), (24), (25), (26) can be presented as

$$\begin{aligned}\tilde{H} &= \tilde{H}_0 + \lambda \tilde{H}_1, & \tilde{P}^k &= \tilde{P}_0^k, \\ \tilde{M}^{k0} &= \tilde{M}_0^{k0} + \lambda \tilde{M}_1^{k0}, & \tilde{M}^{kl} &= \tilde{M}_0^{kl}.\end{aligned}\tag{27}$$

For $\lambda = 0$ - case, the check of relations (22) is identical to the standard check of the Poincare invariance of the free quantum field theory. For general case, it is sufficiently to justify the following commutation relations:

$$[\tilde{H}_1, \tilde{P}_0^k] = 0, \quad [\tilde{H}_1, \tilde{M}_0^{kl}] = 0,\tag{28}$$

$$[\tilde{M}_1^{k0}, \tilde{M}_1^{l0}] = 0, \quad [\tilde{M}_1^{k0}, \tilde{H}_1] = 0, \quad (30)$$

$$[\tilde{M}_1^{k0}, \tilde{H}_0] + [\tilde{M}_0^{k0}, \tilde{H}_1] = 0, \quad [\tilde{M}_1^{k0}, \tilde{M}_0^{l0}] + [\tilde{M}_0^{k0}, \tilde{M}_1^{l0}] = 0. \quad (31)$$

It is straightforward to check that

$$[\tilde{P}_0^l, Q_0(\mathbf{x})] = -i\partial^l Q_0(\mathbf{x}), \quad [\tilde{M}_0^{mn}, Q_0(\mathbf{x})] = -i(x^m \partial^n - x^n \partial^m) Q_0(\mathbf{x}).$$

We obtain relations (28) and (29) then. Eqs. (30) are corollaries of the property $[Q(\mathbf{x}), Q(\mathbf{y})] = 0$. The relation $[\mathcal{E}(\mathbf{x}), Q_0(\mathbf{y})] \sim \delta(\mathbf{x} - \mathbf{y})$ imply eq.(31). Thus, the formal Poincare invariance is checked. However, the divergences and renormalization have not been considered yet.

2.5 Mode decomposition

We have specified states of the $N = \infty$ -theory as sets

$$f = \begin{pmatrix} f_0 \\ f_1[\varphi^1(\cdot)] \\ \dots \\ f_k[\varphi^1(\cdot), \dots, \varphi^k(\cdot)] \\ \dots \end{pmatrix} \quad (32)$$

of symmetric functionals $f_k[\varphi^1, \dots, \varphi^k]$ satisfying relation (5) such that

$$\|f\|^2 = \sum_{k=0}^{\infty} \int D\varphi^1 \dots D\varphi^k |f[\varphi^1(\cdot), \dots, \varphi^k(\cdot)]|^2 < \infty.$$

However, this definition is ill-defined since the measure of functional integration is not determined mathematically. Instead of constructing the measure, it is convenient to use another representation for the k -field functionals.

Consider the basis functionals

$$\Phi_{\mathbf{k}_1 \dots \mathbf{k}_n}^{(n)}[\varphi(\cdot)] = \frac{1}{\sqrt{n!}} a_{\mathbf{k}_1}^+ \dots a_{\mathbf{k}_n}^+ \Phi_0[\varphi(\cdot)], \quad n = 1, 2, 3, \dots \quad (33)$$

corresponding to n particles with momenta $\mathbf{k}_1, \dots, \mathbf{k}_n$. The operators $a_{\mathbf{k}}^+$ are usual quantum field creation operators:

$$a_{\mathbf{k}}^+ = \frac{1}{(2\pi)^{d/2}} \int d\mathbf{x} e^{i\mathbf{k}\mathbf{x}} \left[\sqrt{\frac{\omega_{\mathbf{k}}}{2}} \varphi(\mathbf{x}) - \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \frac{\delta}{\delta \varphi(\mathbf{x})} \right]$$

with $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu^2}$. Integrating by parts and using the commutation relations between creation and annihilation operators, we find that the inner product (1) for the functionals (33) has the form

$$\begin{aligned} (\Phi_{\mathbf{k}_1 \dots \mathbf{k}_n}^{(n)}, \Phi_{\mathbf{p}_1 \dots \mathbf{p}_m}^{(m)}) &= 0, \quad m \neq n; \\ (\Phi_{\mathbf{k}_1 \dots \mathbf{k}_n}^{(n)}, \Phi_{\mathbf{p}_1 \dots \mathbf{p}_n}^{(n)}) &= \frac{1}{n!} \sum_{\sigma} \delta(\mathbf{k}_1 - \mathbf{p}_{\sigma_1}) \dots \delta(\mathbf{k}_n - \mathbf{p}_{\sigma_n}), \end{aligned} \quad (34)$$

the sum is taken over all transpositions of indices $1, \dots, n$. Eqs.(34) can be viewed as a definition of the functional integral (1).

Decompose the functional f_k satisfying eq.(5) as

$$f_k(\varphi^1(\cdot), \dots, \varphi^k(\cdot)) = \sum_{l_1 \dots l_k=1}^{\infty} \int d\mathbf{p}_1^1 \dots d\mathbf{p}_{l_1}^1 \dots d\mathbf{p}_1^k \dots d\mathbf{p}_{l_k}^k f_{l_1; \mathbf{p}_1^1 \dots \mathbf{p}_{l_1}^1; \dots; l_k; \mathbf{p}_1^k \dots \mathbf{p}_{l_k}^k}^{(l_1)} [\phi^1(\cdot)] \dots \Phi_{\mathbf{p}_1^k \dots \mathbf{p}_{l_k}^k}^{(l_k)} [\phi^k(\cdot)].$$

$$f_{l_1; \mathbf{p}_1^1 \dots \mathbf{p}_{l_1}^1, \dots, l_k, \mathbf{p}_1^k \dots \mathbf{p}_{l_k}^k}^k \quad (35)$$

being symmetric under transpositions of \mathbf{p}_i^m and \mathbf{p}_j^m , as well as under transpositions of sets $l_m, \mathbf{p}_1^m \dots \mathbf{p}_{l_m}^m$ and $l_s, \mathbf{p}_1^s \dots \mathbf{p}_{l_s}^s$. The quantity $\|f\|^2$ can be presented as

$$\|f\|^2 = \sum_{k=0}^{\infty} \sum_{l_1 \dots l_k=1}^{\infty} \int d\mathbf{p}_1^1 \dots d\mathbf{p}_{l_1}^1 \dots d\mathbf{p}_1^k \dots d\mathbf{p}_{l_k}^k |f_{l_1; \mathbf{p}_1^1 \dots \mathbf{p}_{l_1}^1, \dots, l_k, \mathbf{p}_1^k \dots \mathbf{p}_{l_k}^k}^k|^2.$$

Creation and annihilation operators can be decomposed as

$$\begin{aligned} A^+[\varphi(\cdot)] &= \sum_{n=0}^{\infty} \int d\mathbf{k}_1 \dots d\mathbf{k}_n \Phi_{\mathbf{k}_1 \dots \mathbf{k}_n}^{(n)*} [\varphi(\cdot)] A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} \cdot \\ A^-[\varphi(\cdot)] &= \sum_{n=0}^{\infty} \int d\mathbf{k}_1 \dots d\mathbf{k}_n \Phi_{\mathbf{k}_1 \dots \mathbf{k}_n}^{(n)} [\varphi(\cdot)] A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} \cdot \end{aligned} \quad (36)$$

The operators $A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{\pm(n)}$ defined as

$$A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} = \int D\phi A^+[\phi(\cdot)] \Phi_{\mathbf{k}_1 \dots \mathbf{k}_n}^{(n)} [\phi(\cdot)], \quad A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} = \int D\phi A^-[\phi(\cdot)] \Phi_{\mathbf{k}_1 \dots \mathbf{k}_n}^{*(n)} [\phi(\cdot)].$$

create (annihilate) the field in the n -particle state with momenta $\mathbf{k}_1, \dots, \mathbf{k}_n$. They are invariant under transpositions of momenta $\mathbf{k}_1, \dots, \mathbf{k}_n$ and obey the ordinary canonical commutation relations:

$$[A_{\mathbf{k}_1 \dots \mathbf{k}_m}^{\pm(m)}, A_{\mathbf{p}_1 \dots \mathbf{p}_n}^{\pm(n)}] = 0, \quad [A_{\mathbf{k}_1 \dots \mathbf{k}_m}^{-(m)}, A_{\mathbf{p}_1 \dots \mathbf{p}_n}^{+(n)}] = 0, m \neq n \quad (37)$$

$$[A_{\mathbf{k}_1 \dots \mathbf{k}_m}^{-(m)}, A_{\mathbf{p}_1 \dots \mathbf{p}_n}^{+(n)}] = \frac{1}{n!} \sum_{\sigma} \delta(\mathbf{k}_1 - \mathbf{p}_{\sigma_1}) \dots \delta(\mathbf{k}_n - \mathbf{p}_{\sigma_n}). \quad (38)$$

Any vector f can be written via creation operators and vacuum state

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \\ \dots \end{pmatrix} \quad (39)$$

as follows:

$$f = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} \sum_{l_1 \dots l_k=1}^{\infty} \int d\mathbf{p}_1^1 \dots d\mathbf{p}_{l_1}^1 \dots d\mathbf{p}_1^k \dots d\mathbf{p}_{l_k}^k f_{l_1; \mathbf{p}_1^1 \dots \mathbf{p}_{l_1}^1, \dots, l_k, \mathbf{p}_1^k \dots \mathbf{p}_{l_k}^k}^k A_{\mathbf{p}_1^1 \dots \mathbf{p}_{l_1}^1}^{+(l_1)} \dots A_{\mathbf{p}_1^k \dots \mathbf{p}_{l_k}^k}^{+(l_k)} |0\rangle.$$

Making use of the quantum field theory formulas

$$\begin{aligned} &: \int d\mathbf{x} \mathcal{E}(\mathbf{x}) := \int d\mathbf{k} \omega_{\mathbf{k}} a_{\mathbf{k}}^+ a_{\mathbf{k}}^-, \\ \phi^2(\mathbf{x}) \Phi_0 &= \frac{1}{(2\pi)^d} \int \frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i(\mathbf{k}+\mathbf{p})\mathbf{x}} a_{\mathbf{k}}^+ a_{\mathbf{p}}^+ \Phi_0, \end{aligned} \quad (40)$$

one transforms expression (21) to the following form:

$$\begin{aligned} \tilde{H} &= \sum_{n=1}^{\infty} \int d\mathbf{k}_1 \dots d\mathbf{k}_n A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} (\omega_{\mathbf{k}_1} + \dots + \omega_{\mathbf{k}_n}) A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} \\ &+ \frac{\lambda}{4} \int d\mathbf{x} \left(\frac{\sqrt{2}}{(2\pi)^d} \int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} (A_{\mathbf{k}_1 \mathbf{k}_2}^{+(2)} e^{-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{x}} + A_{\mathbf{k}_1 \mathbf{k}_2}^{-(2)} e^{i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{x}}) \right)^2. \end{aligned} \quad (41)$$

$$\begin{aligned}
\tilde{P}^l &= \sum_{n=1}^{\infty} \int d\mathbf{k}_1 \dots d\mathbf{k}_n A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} (k_1^l + \dots + k_n^l) A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} \\
\tilde{M}^{ml} &= \sum_{n=1}^{\infty} \int d\mathbf{k}_1 \dots d\mathbf{k}_n A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} \sum_{s=1}^n (k_s^l i \frac{\partial}{\partial k_s^m} - k_s^m i \frac{\partial}{\partial k_s^l}) A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} \\
\tilde{M}^{l0} &= \sum_{n=1}^{\infty} \int d\mathbf{k}_1 \dots d\mathbf{k}_n A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} \sum_{s=1}^n (i\omega_{\mathbf{k}_s} \frac{\partial}{\partial k_s^l} + i \frac{k_s^l}{2\omega_{\mathbf{k}_s}}) A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} \\
&+ \frac{\lambda}{4} \int d\mathbf{x} x^l \left(\frac{\sqrt{2}}{(2\pi)^d} \int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} (A_{\mathbf{k}_1 \mathbf{k}_2}^{+(2)} e^{-i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}} + A_{\mathbf{k}_1 \mathbf{k}_2}^{-(2)} e^{i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}}) \right)^2.
\end{aligned} \tag{42}$$

2.6 Decomposition of the state space

We see that the Hilbert space of the $N = \infty$ -theory can be presented as

$$\mathcal{F}(\oplus_{n=1}^{\infty} \mathcal{H}^{\vee n}) \tag{43}$$

(the notations of Appendix A are used), where \mathcal{H} is a space of complex functions $f_{\mathbf{k}}$, $\mathbf{k} \in \mathbf{R}^d$ from $L^2(\mathbf{R}^d)$. Analogously to lemma A.8, the space (43) is isomorphic to

$$\mathcal{F}(\mathcal{H}^{\vee 2}) \otimes \mathcal{F}(\mathcal{H} + \oplus_{n=3}^{\infty} \mathcal{H}^{\vee n}) \equiv \mathcal{F} \otimes \check{\mathcal{F}}, \tag{44}$$

while the operators (41) and (42) can be viewed as the following operators in the space (44):

$$\begin{aligned}
\tilde{H} &= H \otimes 1 + 1 \otimes \check{H}, & \tilde{P}^k &= P^k \otimes 1 + 1 \otimes \check{P}^k, \\
\tilde{M}^{ml} &= M^{ml} \otimes 1 + 1 \otimes \check{M}^{ml}, & \tilde{M}^{k0} &= M^{k0} \otimes 1 + 1 \otimes \check{M}^{k0}.
\end{aligned} \tag{45}$$

The operators \check{H} , \check{P}^k , \check{M}^{kl} and \check{M}^{k0} are the same as in the free theory:

$$\begin{aligned}
\check{H} &= \sum_{n=1,3,4,\dots} \int d\mathbf{k}_1 \dots d\mathbf{k}_n A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} (\omega_{\mathbf{k}_1} + \dots + \omega_{\mathbf{k}_n}) A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} \\
\check{P}^l &= \sum_{n=1,3,4,\dots} \int d\mathbf{k}_1 \dots d\mathbf{k}_n A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} (k_1^l + \dots + k_n^l) A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} \\
\check{M}^{ml} &= \sum_{n=1,3,4,\dots} \int d\mathbf{k}_1 \dots d\mathbf{k}_n A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} \sum_{s=1}^n (k_s^l i \frac{\partial}{\partial k_s^m} - k_s^m i \frac{\partial}{\partial k_s^l}) A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)} \\
\check{M}^{l0} &= \sum_{n=1,3,4,\dots} \int d\mathbf{k}_1 \dots d\mathbf{k}_n A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{+(n)} \sum_{s=1}^n (i\omega_{\mathbf{k}_s} \frac{\partial}{\partial k_s^l} + i \frac{k_s^l}{2\omega_{\mathbf{k}_s}}) A_{\mathbf{k}_1 \dots \mathbf{k}_n}^{-(n)}.
\end{aligned} \tag{46}$$

The only nontrivial part of operators (45) correspond to the space $\mathcal{F}(\mathcal{H}^{\vee 2})$:

$$\begin{aligned}
H &= \int d\mathbf{k}_1 d\mathbf{k}_2 A_{\mathbf{k}_1 \mathbf{k}_2}^{+(2)} (\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}) A_{\mathbf{k}_1 \mathbf{k}_2}^{-(2)} \\
&+ \frac{\lambda}{4} \int d\mathbf{x} \left(\frac{\sqrt{2}}{(2\pi)^d} \int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} (A_{\mathbf{k}_1 \mathbf{k}_2}^{+(2)} e^{-i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}} + A_{\mathbf{k}_1 \mathbf{k}_2}^{-(2)} e^{i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}}) \right)^2 \\
\tilde{P}^l &= \int d\mathbf{k}_1 d\mathbf{k}_2 A_{\mathbf{k}_1 \mathbf{k}_2}^{+(2)} (k_1^l + k_2^l) A_{\mathbf{k}_1 \mathbf{k}_2}^{-(2)} \\
M^{ml} &= \int d\mathbf{k}_1 d\mathbf{k}_2 A_{\mathbf{k}_1 \mathbf{k}_2}^{+(2)} \sum_{s=1}^2 (k_s^l i \frac{\partial}{\partial k_s^m} - k_s^m i \frac{\partial}{\partial k_s^l}) A_{\mathbf{k}_1 \mathbf{k}_2}^{-(2)} \\
M^{l0} &= \int d\mathbf{k}_1 d\mathbf{k}_2 A_{\mathbf{k}_1 \mathbf{k}_2}^{+(2)} \sum_{s=1}^2 (i\omega_{\mathbf{k}_s} \frac{\partial}{\partial k_s^l} + i \frac{k_s^l}{2\omega_{\mathbf{k}_s}}) A_{\mathbf{k}_1 \mathbf{k}_2}^{-(2)} \\
&+ \frac{\lambda}{4} \int d\mathbf{x} x^l \left(\frac{\sqrt{2}}{(2\pi)^d} \int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} (A_{\mathbf{k}_1 \mathbf{k}_2}^{+(2)} e^{-i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}} + A_{\mathbf{k}_1 \mathbf{k}_2}^{-(2)} e^{i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}}) \right)^2.
\end{aligned} \tag{47}$$

$$\tilde{U}_{\Lambda,a} = U_{\Lambda,a} \otimes \check{U}_{\Lambda,a}$$

with

$$U_{\Lambda,a} = \exp(iP_\mu a^\mu) \exp\left(\frac{i}{2} M^{\Lambda\mu} \theta_{\lambda\mu}\right), \quad \check{U}_{\Lambda,a} = \exp(i\check{P}_\mu a^\mu) \exp\left(\frac{i}{2} \check{M}^{\lambda\mu} \theta_{\lambda\mu}\right)$$

To express the operators $\check{U}_{\Lambda,a}$, it is convenient to introduce the operators $u_{\Lambda,a}$ of the unitary representation of the Poincare group in \mathcal{H} :

$$(u_{\Lambda,a} f)_{\mathbf{k}} = \exp(i\omega_{\mathbf{k}} a_0 - i\mathbf{k}\mathbf{a}) \sqrt{\frac{(\Lambda^{-1})_n^0 k^n + (\Lambda^{-1})_0^0 \omega_{\mathbf{k}}}{\omega_{\mathbf{k}}}} f_{(\Lambda^{-1})_n^m k^n + (\Lambda^{-1})_0^m \omega_{\mathbf{k}}} \quad (48)$$

with generators

$$\begin{aligned} p^l &= k^l, & p^0 &= \omega_{\mathbf{k}} & m^{l0} &= i(\omega_{\mathbf{k}} \frac{\partial}{\partial k^l} + \frac{k^l}{2\omega_{\mathbf{k}}}), \\ m^{ln} &= i(k^n \frac{\partial}{\partial k^l} - k^l \frac{\partial}{\partial k^n}). \end{aligned} \quad (49)$$

By $\tilde{u}_{\Lambda,a} : \mathcal{H} \oplus \oplus_{n=3}^{\infty} \mathcal{H}^{\vee n} \rightarrow \mathcal{H} \oplus \oplus_{n=3}^{\infty} \mathcal{H}^{\vee n}$ we denote the operator

$$\tilde{u}_{\Lambda,a}(f_1, f_3, f_4, \dots) = (u_{\Lambda,a} f_1, u_{\Lambda,a}^{\otimes 3} f_3, u_{\Lambda,a}^{\otimes 4} f_4, \dots).$$

We can notice that

$$\check{U}_{\Lambda,a} = \mathcal{U}(\tilde{u}_{\Lambda,a})$$

(the notations of Appendix A are used).

Thus, the operators $\check{U}_{\Lambda,a}$ are constructed. The only nontrivial problem is to construct the representation of the Poincare group corresponding to the generators (47).

2.7 Problem of divergences

2.7.1 The Haag theorem and volume divergences

Apply the Hamiltonian (47) to the vacuum state. The result will be

$$H|0\rangle = \frac{\lambda}{4} \frac{2}{(2\pi)^d} \int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} \frac{d\mathbf{p}_1}{\sqrt{2\omega_{\mathbf{p}_1}}} \frac{d\mathbf{p}_2}{\sqrt{2\omega_{\mathbf{p}_2}}} A_{\mathbf{k}_1\mathbf{k}_2}^+ A_{\mathbf{p}_1\mathbf{p}_2}^+ \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{p}_1 + \mathbf{p}_2) |0\rangle,$$

where $A_{\mathbf{k}_1\mathbf{k}_2}^+ \equiv A_{\mathbf{k}_1\mathbf{k}_2}^{+(2)}$. Because of the δ -function, the quantity $\|H\Phi^{(0)}\|^2$ diverges. This is a volume divergence associated with the Haag theorem (see, for example, [6]). An analogous infinite quantity appears when one applied the perturbation theory in λ for the evolution operator.

Within the perturbation theory, such difficulty can be resolved with the help of the Faddeev transformation [12].

2.7.2 The Stueckelberg divergences

Even after removing the vacuum divergences, the problem is not completely resolved. If one considers the perturbation theory for the Schrodinger equation of motion, one finds that there are UV-divergences even in the tree approximation. Namely, for the first order of the perturbation theory one has

$$\begin{aligned} e^{iH_0 t} \tilde{H}_1 e^{-iH_0 t} \equiv \tilde{H}_1(t) &= \frac{1}{(2\pi)^d} \int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} \frac{d\mathbf{p}_1}{\sqrt{2\omega_{\mathbf{p}_1}}} \frac{d\mathbf{p}_2}{\sqrt{2\omega_{\mathbf{p}_2}}} A_{\mathbf{k}_1\mathbf{k}_2}^+ A_{\mathbf{p}_1\mathbf{p}_2}^- \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_1 - \mathbf{p}_2) \\ &\quad \times e^{-it(\omega_{\mathbf{k}_1} \omega_{\mathbf{k}_2} + \omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2})}. \end{aligned} \quad (50)$$

$$U_t = -i\lambda \int_0^t d\tau \tilde{H}_1(\tau)$$

to the vector

$$\Phi^0 = \int d\mathbf{p}_1 d\mathbf{p}_2 A_{\mathbf{p}_1 \mathbf{p}_2}^+ \Phi_{\mathbf{p}_1 \mathbf{p}_2}^0,$$

we find

$$U_t \Phi^0 = \int d\mathbf{p}_1 d\mathbf{p}_2 A_{\mathbf{p}_1 \mathbf{p}_2}^+ \Phi_{\mathbf{p}_1 \mathbf{p}_2}^t$$

with

$$\Phi_{\mathbf{k}_1 \mathbf{k}_2}^t = \frac{1}{(2\pi)^d} \frac{1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}_2}}} \int \frac{1}{\sqrt{2\omega_{\mathbf{p}_1}}} \frac{1}{\sqrt{2\omega_{\mathbf{p}_2}}} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_1 - \mathbf{p}_2) \Phi_{\mathbf{p}_1 \mathbf{p}_2}^{(0)} \frac{e^{-it(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2})} - 1}{i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2})}.$$

The integral

$$\int d\mathbf{k}_1 d\mathbf{k}_2 |\Phi_{\mathbf{k}_1 \mathbf{k}_2}^t|^2$$

diverges for $d \geq 4$. This is a Stueckelberg divergence.

3 Construction of the Hamiltonian

The purpose of this section is to define mathematically the operators in the Hilbert space that corresponds to the formal expression (47). For $d+1 = 4, 5$, it is necessary to perform the infinite renormalization of the coupling constant, for $d+1 \geq 6$, the model is nonrenormalizable. Subsections A-C deal with heuristic construction of the Hamiltonian; in subsection D representation for space-time translations and spatial rotations is constructed.

3.1 Diagonalization of the Hamiltonian

Since the Hamiltonian (47) is quadratic with respect to creation and annihilation operators, one can perform the canonical transformation of creation and annihilation operators in order to take the Hamiltonian to the canonical form.

It is convenient to introduce new variables,

$$\begin{aligned} Q_{\mathbf{P}\mathbf{s}} &= \frac{1}{\sqrt{2\Omega_{\mathbf{P}\mathbf{s}}}} (A_{\mathbf{P}/2-\mathbf{s}, \mathbf{P}/2+\mathbf{s}}^+ + A_{-\mathbf{P}/2-\mathbf{s}, -\mathbf{P}/2+\mathbf{s}}^-), \\ \Pi_{\mathbf{P}\mathbf{s}} &= i\sqrt{\frac{\Omega_{\mathbf{P}\mathbf{s}}}{2}} (A_{\mathbf{P}/2-\mathbf{s}, \mathbf{P}/2+\mathbf{s}}^+ - A_{-\mathbf{P}/2-\mathbf{s}, -\mathbf{P}/2+\mathbf{s}}^-). \end{aligned} \quad (51)$$

Here

$$\Omega_{\mathbf{P}\mathbf{s}} = \omega_{\mathbf{P}/2-\mathbf{s}} + \omega_{\mathbf{P}/2+\mathbf{s}}. \quad (52)$$

The operators (51) obey the properties:

$$Q_{\mathbf{P}\mathbf{s}} = Q_{\mathbf{P}, -\mathbf{s}} = Q_{-\mathbf{P}, \mathbf{s}}^*, \quad \Pi_{\mathbf{P}\mathbf{s}} = \Pi_{\mathbf{P}, -\mathbf{s}} = \Pi_{-\mathbf{P}, \mathbf{s}}^*$$

and canonical commutation relations:

$$[Q_{\mathbf{P}\mathbf{s}}, Q_{\mathbf{P}'\mathbf{s}'}] = 0, \quad [\Pi_{\mathbf{P}\mathbf{s}}, \Pi_{\mathbf{P}'\mathbf{s}'}] = 0, \quad [Q_{\mathbf{P}\mathbf{s}}, \Pi_{\mathbf{P}'\mathbf{s}'}] = i\delta_{\mathbf{P}\mathbf{P}'} \frac{1}{2} (\delta_{\mathbf{s}-\mathbf{s}'} + \delta_{\mathbf{s}+\mathbf{s}'}).$$

$$H = \frac{1}{2} \int d\mathbf{P} (\Pi_{\mathbf{P}}, \Pi_{\mathbf{P}}) + \frac{1}{2} \int d\mathbf{P} (Q_{\mathbf{P}}, (M_{\mathbf{P}})^2 Q_{\mathbf{P}}). \quad (53)$$

Here $\Pi_{\mathbf{P}}$, $Q_{\mathbf{P}}$ are operator-valued even functions of the variable \mathbf{s} . The inner product (f, g) of two even functions $f_{\mathbf{s}}$ and $g_{\mathbf{s}}$ is, as usual, $\int d\mathbf{s} f_{\mathbf{s}}^* g_{\mathbf{s}}$. $(M_{\mathbf{P}})^2$ is the following operator in the space of even functions:

$$((M_{\mathbf{P}})^2 \varphi)_{\mathbf{s}} = \Omega_{\mathbf{P}\mathbf{s}}^2 \varphi_{\mathbf{s}} + \frac{\lambda}{(2\pi)^d} \int d\mathbf{s}' \sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}} 2\omega_{\mathbf{P}/2-\mathbf{s}}}} \sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}'}}{2\omega_{\mathbf{P}/2+\mathbf{s}'} 2\omega_{\mathbf{P}/2-\mathbf{s}'}}} \varphi_{\mathbf{s}'} \quad (54)$$

The operator (53) can be diagonalized by the following procedure:

$$\begin{aligned} Q_{\mathbf{P}} &= \frac{1}{\sqrt{2M_{\mathbf{P}}}} (C_{\mathbf{P}}^+ + C_{-\mathbf{P}}^-), \\ \Pi_{\mathbf{P}} &= i \sqrt{\frac{M_{\mathbf{P}}}{2}} (C_{\mathbf{P}}^+ - C_{-\mathbf{P}}^-). \end{aligned} \quad (55)$$

where $C_{\mathbf{P}}^{\pm}$ are operator-valued functions $C_{\mathbf{P}\mathbf{s}}^{\pm}$ of the variable \mathbf{s} . They obey the usual canonical commutation relations:

$$[C_{\mathbf{P}\mathbf{s}}^-, C_{\mathbf{P}\mathbf{s}'}^+] = \delta_{\mathbf{P}\mathbf{P}'} \frac{1}{2} (\delta_{\mathbf{s}-\mathbf{s}'} + \delta_{\mathbf{s}+\mathbf{s}'}), \quad [C_{\mathbf{P}\mathbf{s}}^{\pm}, C_{\mathbf{P}\mathbf{s}}^{\pm}] = 0.$$

The Hamiltonian takes the form:

$$H = \int d\mathbf{P} d\mathbf{s} d\mathbf{s}' C_{\mathbf{P}\mathbf{s}}^+ (M_{\mathbf{P}})_{\mathbf{s}\mathbf{s}'} C_{\mathbf{P}\mathbf{s}'}^-. \quad (56)$$

The $(M_{\mathbf{P}})_{\mathbf{s}\mathbf{s}'}$ is a matrix element of the operator $M_{\mathbf{P}}$.

One should use then another, non-Fock representation for the operators $A_{\mathbf{k}_1 \mathbf{k}_2}^{\pm}$, which is Fock representation for the transformed operators $C_{\mathbf{P}\mathbf{s}}^{\pm}$. The Hamiltonian (56) is then a self-adjoint operator if $M_{\mathbf{P}}$ is self-adjoint. The evolution operator is expressed via the unitary operator $e^{-iM_{\mathbf{P}}t}$. To construct $M_{\mathbf{P}}$, one should first check that $(M_{\mathbf{P}})^2$ is a positively definite self-adjoint operator and define $M_{\mathbf{P}} \equiv \sqrt{(M_{\mathbf{P}})^2}$, making use of the functional calculus of self-adjoint operators. The operator $(M_{\mathbf{P}})^{1/2}$ entering to expression (55) can be constructed in analogous way.

3.2 Definition of the Hamiltonian and its properties

Formula (54) for the operator $(M_{\mathbf{P}})^2$ is not well-defined since the vector

$$\sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}'} 2\omega_{\mathbf{P}/2-\mathbf{s}'}}}$$

considered as a function of \mathbf{s} does not belong to L^2 . The operator (54) is therefore analogous to the quantum mechanical Hamiltonian corresponding to the particle moving in the singular potential like δ -function. The theory of singular potentials is developed in [14].

To construct mathematically the (unbounded) self-adjoint operator $(M_{\mathbf{P}})^2$ one may first construct the (bounded) operator $(M_{\mathbf{P}})^{-2}$, prove that it is invertible and positively definite. Then the operator $(M_{\mathbf{P}})^2$ is defined as $(M_{\mathbf{P}})^2 \equiv ((M_{\mathbf{P}})^{-2})^{-1}$.

To find the vector

$$\varphi = (M_{\mathbf{P}})^{-2} \psi,$$

$$\begin{aligned}\psi_{\mathbf{s}} &= \Omega_{\mathbf{P}\mathbf{s}}^2 \varphi_{\mathbf{s}} + c \sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}}2\omega_{\mathbf{P}/2-\mathbf{s}}}}, \\ c &= \frac{\lambda}{(2\pi)^d} \int d\mathbf{s} \sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}}2\omega_{\mathbf{P}/2-\mathbf{s}}}} \varphi_{\mathbf{s}}.\end{aligned}\tag{57}$$

Eqs. (57) imply that

$$c = \frac{\lambda_R^{\mathbf{P}}}{(2\pi)^d} \int d\mathbf{s} \sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}}2\omega_{\mathbf{P}/2-\mathbf{s}}}} \frac{1}{\Omega_{\mathbf{P}\mathbf{s}}^2} \psi_{\mathbf{s}}\tag{58}$$

with the "renormalized" coupling constant $\lambda_R^{\mathbf{P}}$ expressed from the relation

$$\frac{1}{\lambda_R^{\mathbf{P}}} = \frac{1}{\lambda} + \frac{1}{(2\pi)^d} \int d\mathbf{s} \frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}}2\omega_{\mathbf{P}/2-\mathbf{s}}} \frac{1}{\Omega_{\mathbf{P}\mathbf{s}}^2}.\tag{59}$$

The operator $(M_{\mathbf{P}})^{-2}$ has then the form

$$((M_{\mathbf{P}})^{-2}\psi)_{\mathbf{s}} = \frac{1}{\Omega_{\mathbf{P}\mathbf{s}}^2} \psi_{\mathbf{s}} - \frac{\lambda_R^{\mathbf{P}}}{(2\pi)^d} \chi_{\mathbf{P}\mathbf{s}} \int d\mathbf{s}' \chi_{\mathbf{P}\mathbf{s}'} \psi_{\mathbf{s}'}\tag{60}$$

with

$$\chi_{\mathbf{P}\mathbf{s}} = \frac{1}{\Omega_{\mathbf{P}\mathbf{s}}^2} \cdot \sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}}2\omega_{\mathbf{P}/2-\mathbf{s}}}}.\tag{61}$$

The function (61) treated as a function of \mathbf{s} belongs to L^2 for $d+1 < 6$. For these values of the space-time dimensionality, the operator $(M_{\mathbf{P}})^{-2}$ is bounded and self-adjoint, provided that the quantity $\lambda_R^{\mathbf{P}}$ is finite. Since the integral entering to the right-hand side of eq.(59) diverges at $d+1 = 4, 5$, for these values of d it is necessary to perform infinite renormalization of the coupling constant. This means that λ should be chosen in such a way that $|\lambda_R^{\mathbf{P}}| < \infty$. The fact that λ is \mathbf{P} -independent means that

$$\frac{1}{\lambda_R^{\mathbf{P}_1}} - \frac{1}{\lambda_R^{\mathbf{P}_2}} = \frac{1}{(2\pi)^d} \int d\mathbf{s} \left[\frac{1}{2\Omega_{\mathbf{P}_1\mathbf{s}}\omega_{\mathbf{P}_1/2+\mathbf{s}}\omega_{\mathbf{P}_1/2-\mathbf{s}}} - \frac{1}{2\Omega_{\mathbf{P}_2\mathbf{s}}\omega_{\mathbf{P}_2/2+\mathbf{s}}\omega_{\mathbf{P}_2/2-\mathbf{s}}} \right]\tag{62}$$

Note that the integral in the right-hand side of eq.(62) is well-defined at $d+1 = 4, 5$, since

$$\frac{\partial}{\partial \mathbf{P}} \left(\frac{1}{2\Omega_{\mathbf{P}_2\mathbf{s}}\omega_{\mathbf{P}_2/2+\mathbf{s}}\omega_{\mathbf{P}_2/2-\mathbf{s}}} \right) = O(|\mathbf{s}|^{-5}), s \rightarrow \infty.$$

The fact that the operator (60) is invertible can be understood as follows. Suppose that $(M_{\mathbf{P}})^{-2}\psi = 0$ for some ψ . This means that

$$\psi_{\mathbf{s}} = c\Omega_{\mathbf{P}\mathbf{s}}^2 \chi_{\mathbf{P}\mathbf{s}}\tag{63}$$

for some multiplier c . But the function (63) does not belong to L^2 . Thus, the operator $(M_{\mathbf{P}})^{-2}$ is invertible.

To investigate the positive definiteness of the operator $(M_{\mathbf{P}})^{-2}$, calculate the integral

$$I(\mathbf{P}, \varepsilon) = \frac{1}{(2\pi)^d} \int \frac{d\mathbf{s}}{2\omega_{\mathbf{P}_2/2+\mathbf{s}}\omega_{\mathbf{P}_2/2-\mathbf{s}}} \frac{1}{\Omega_{\mathbf{P}_2\mathbf{s}}^2 + \varepsilon^2},$$

making use of the dimensional regularization. First of all, introduce new variables, $\mathbf{k}_1 = \mathbf{P}/2 + \mathbf{s}$, $\mathbf{k}_2 = \mathbf{P}/2 - \mathbf{s}$, so that $\int d\mathbf{s} \rightarrow \int d\mathbf{k}_1 d\mathbf{k}_2 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{P})$. Next, use the identity

$$\frac{\omega_1 + \omega_2}{2\omega_1\omega_2(\varepsilon^2 + (\omega_1 + \omega_2)^2)} = \frac{1}{2\pi} \int \frac{d\xi}{(\omega_1^2 + \xi^2)(\omega_2^2 + (\xi - \varepsilon)^2)},$$

$$I(\mathbf{P}, \varepsilon) = \frac{1}{(2\pi)^{d+1}} \int \frac{\alpha \mathbf{k}_1 \alpha \mathbf{k}_2 \alpha \xi \alpha (\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{P})}{(\xi^2 + \omega_{\mathbf{k}_1}^2)((\xi - \varepsilon)^2 + \omega_{\mathbf{k}_2}^2)}.$$

Introduce, as usual, the α -representation: $a^{-1} = \int_0^\infty d\alpha e^{-\alpha a}$. We get

$$I(\mathbf{P}, \varepsilon) = \frac{1}{(2\pi)^{d+1}} \int_0^\infty d\alpha \int_0^\infty d\beta e^{-\mu^2(\alpha+\beta)} \left(\frac{\pi}{\alpha + \beta} \right)^{\frac{d+1}{2}} e^{-\frac{\alpha\beta}{\alpha+\beta}(\mathbf{P}^2 + \varepsilon^2)}. \quad (64)$$

Therefore,

$$\frac{1}{\lambda_R^{\mathbf{P}}} - \frac{1}{\lambda_R^0} = \frac{1}{(2\pi)^{d+1}} \int_0^\infty d\alpha \int_0^\infty d\beta e^{-\mu^2(\alpha+\beta)} \left(\frac{\pi}{\alpha + \beta} \right)^{\frac{d+1}{2}} (e^{-\frac{\alpha\beta}{\alpha+\beta}\mathbf{P}^2} - 1) < 0. \quad (65)$$

The requirement $(M_{\mathbf{P}})^{-2} \geq 0$ is a corollary of the condition $\lambda_R^{\mathbf{P}} < 0$. Inequality (65) implies that it is sufficient to require $\lambda_R < 0$. This is a well-known condition of absence of tachyons [15] in the large- N theory.

Thus, we have constructed the Hamiltonian.

Another way to define the Hamiltonian is the following [14]. One can use the theory of self-adjoint extensions [16]. Consider the operator $\Omega_{\mathbf{P}\mathbf{s}}^2$ defined on the domain consisting of such φ that

$$\int d\mathbf{s} \varphi_{\mathbf{s}} \sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}} 2\omega_{\mathbf{P}/2-\mathbf{s}}}} = 0.$$

If $d+1 \geq 6$, the operator is essentially self-adjoint. This corresponds to the "triviality" (or nonrenormalizability) of the model. If $d+1 < 6$, there is a one-parametric family of self-adjoint extensions specified by the parameter $\lambda_R^{\mathbf{P}}$. However, the condition (62) cannot be obtained by the self-adjoint extension method. One should use another argumentation like Poincare invariance.

3.3 Momentum and angular momentum

Let us express the momentum and angular momentum operators via new creation and annihilation operators $C_{\mathbf{P}\mathbf{s}}^\pm$. It follows from eqs.(51) that operators (47) can be written as

$$P^l = \int d\mathbf{P} d\mathbf{s} Q_{\mathbf{P}\mathbf{s}} P^l i\Pi_{-\mathbf{P}\mathbf{s}},$$

$$M^{ml} = \int d\mathbf{P} d\mathbf{s} Q_{\mathbf{P}\mathbf{s}} (iP^l \frac{\partial}{\partial P^m} + is^l \frac{\partial}{\partial s^m} - iP^m \frac{\partial}{\partial P^l} - is^m \frac{\partial}{\partial s^l}) i\Pi_{-\mathbf{P}\mathbf{s}}. \quad (66)$$

Since the kernel of the operator $M_{\mathbf{P}}^{-2}$ (60) is invariant under spatial rotations

$$\mathbf{P} \rightarrow O\mathbf{P}, \quad \mathbf{s} \rightarrow O\mathbf{s}, \quad \mathbf{s}' \rightarrow O\mathbf{s}'$$

with orthogonal matrix O , it commutes with the rotation operator of the form $O f_{\mathbf{P}\mathbf{s}} = f_{O\mathbf{P}, O\mathbf{s}}$. Analogously, any function of M obey this property. Since the operator

$$iP^l \frac{\partial}{\partial P^m} + is^l \frac{\partial}{\partial s^m} - iP^m \frac{\partial}{\partial P^l} - is^m \frac{\partial}{\partial s^l}$$

is a generator of a rotation, it commutes with any function of M . Making use of this property, we find

$$M^{ml} = \int d\mathbf{P} d\mathbf{s} C_{\mathbf{P}\mathbf{s}}^+ (iP^l \frac{\partial}{\partial P^m} + is^l \frac{\partial}{\partial s^m} - iP^m \frac{\partial}{\partial P^l} - is^m \frac{\partial}{\partial s^l}) C_{\mathbf{P}\mathbf{s}}^-.$$

Analogously,

$$P^l = \int d\mathbf{P} d\mathbf{s} C_{\mathbf{P}\mathbf{s}}^+ P^l C_{\mathbf{P}\mathbf{s}}^-.$$

The problem of divergences made us to change the representation for the operators $A_{\mathbf{k}_1\mathbf{k}_2}^{\pm(2)}$. We have considered the space $\mathcal{H}_2 \subset L^2(\mathbf{R}^{2d})$ of functions $f_{\mathbf{P}\mathbf{s}}$ which obey the property $f_{\mathbf{P},-\mathbf{s}} = f_{\mathbf{P},\mathbf{s}}$. The space $\mathcal{F}(\mathcal{H}_2)$ has been considered instead of $\mathcal{F}(\mathcal{H}^{\vee 2})$. Therefore, the space (44) is substituted by $\mathcal{F}(\mathcal{H}_2) \otimes \check{\mathcal{F}}$. One should define then operators H , P^l , M^{ml} , M^{m0} and $U_{\Lambda,a}$ in $\mathcal{F}(\mathcal{H}_2)$ that corresponds to formal expressions (47).

The operators H , P^l , M^{ml} have been considered above.

Let λ_R^0 be a fixed negative quantity. Set $\mathbf{P}_2 = 0$, $\mathbf{P}_1 = \mathbf{P}$ in eq.(62) and define the quantity $\lambda_R^{\mathbf{P}}$. Consider the operator $M^{-2} : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ of the form

$$(M^{-2}\psi)_{\mathbf{P}\mathbf{s}} = \Omega_{\mathbf{P}\mathbf{s}}^{-2}\psi_{\mathbf{P}\mathbf{s}} - \frac{\lambda_R^{\mathbf{P}}}{(2\pi)^d}\chi_{\mathbf{P}\mathbf{s}} \int ds' \chi_{\mathbf{P}\mathbf{s}'} \psi_{\mathbf{P}\mathbf{s}'},$$

where $\chi_{\mathbf{P}\mathbf{s}}$ has the form (61). Since the operator M^{-2} is positively definite and self-adjoint, the self-adjoint positive operator $M \equiv (M^{-2})^{-1/2}$ is uniquely defined. The Hamiltonian operator is

$$H = \mathcal{F}(M)$$

(the notations from appendix A are used), while

$$e^{-iHt} = \mathcal{U}(e^{-iMt})$$

Analogously,

$$P^l = \mathcal{F}(P^l),$$

$$M^{ml} = \mathcal{F}(iP^l \frac{\partial}{\partial P^m} + is^l \frac{\partial}{\partial s^m} - iP^m \frac{\partial}{\partial P^l} - is^m \frac{\partial}{\partial s^l}).$$

The space rotations being Lorentz transformations with

$$\Lambda_i^0 = 0, \quad \Lambda_0^i = 0, \quad \Lambda_0^0 = 1 \quad (67)$$

are represented by the operators $U_{\Lambda,0} = \mathcal{U}(u_{\Lambda,0})$ with

$$(u_{\Lambda,0}f)_{\mathbf{P}\mathbf{s}} = f_{\Lambda^{-1}\mathbf{P},\Lambda^{-1}\mathbf{s}}. \quad (68)$$

For space-time translations, one has

$$U_{1,a} = e^{iHt} e^{-iP^l a^l} = \mathcal{U}(e^{iMt} e^{-i\mathbf{P}\mathbf{a}}). \quad (69)$$

Thus, we have constructed the operators $U_{\Lambda,a}$ corresponding to the Poincare transformations obeying eq.(67): $U_{\Lambda,a} = U_{0,a} U_{\Lambda,0}$.

Lemma 3.1. *The group property*

$$U_{\Lambda_1,a_1} U_{\Lambda_2,a_2} = U_{\Lambda_1\Lambda_2,a_1+\Lambda_1a_2}$$

is satisfied for Poincare transformations obeying eq.(67).

Proof. It is sufficient to show that

$$U_{\Lambda_1\Lambda_2,0} = U_{\Lambda_1,0} U_{\Lambda_2,0} \quad (70)$$

$$U_{1,a_1} U_{1,a_2} = U_{1,a_1+a_2}, \quad (71)$$

$$U_{\Lambda,0} U_{1,a} U_{\Lambda,0}^{-1} = U_{1,\Lambda a}. \quad (72)$$

$$[e^{iMt}, e^{-i\mathbf{P}\mathbf{a}}] = 0. \quad (73)$$

Definition (68) and commutation relation

$$[u_{\Lambda,0}, e^{iMt}] = 0 \quad (74)$$

imply property (72). Lemma 3.1 is proved.

Let us check now some axioms of quantum field theory.

Lemma 3.2. (Existence and uniqueness of vacuum). *For vector $\Phi \in \mathcal{F}(\mathcal{H}_2)$ the following statements are equivalent:*

(i) *invariance under space-time translations: for all a*

$$U_{0,a}\Phi = \Phi;$$

(ii) $\Phi = c|0\rangle$ for some multiplier $c \in \mathbf{C}$.

The proof is obvious.

Investigate now spectral properties.

Lemma 3.3. *The spectrum of the operator P^μ is a subset of a set $\{0\} \cup \{(\varepsilon, \mathbf{p}) | \varepsilon^2 - \mathbf{p}^2 > 0\}$.*

Proof. It is sufficient to prove that $\sigma(M_{\mathbf{P}}^2) \subset (\mathbf{P}^2, \infty)$. The property $\varepsilon^2 \in \sigma(M_{\mathbf{P}}^2)$ means that the operator $\varepsilon^2 - M_{\mathbf{P}}^2$ is not boundedly invertible. Since

$$((\varepsilon^2 - M_{\mathbf{P}}^2)^{-1}\psi)_{\mathbf{s}} = (\varepsilon^2 - \Omega_{\mathbf{P}\mathbf{s}}^2)^{-1}\psi_{\mathbf{s}} + \frac{(\Omega_{\mathbf{P}\mathbf{s}}^2 - \varepsilon^2)^{-1}\Omega_{\mathbf{P}\mathbf{s}}^2\chi_{\mathbf{P}\mathbf{s}} \int d\mathbf{s}' \chi_{\mathbf{P}\mathbf{s}'} \Omega_{\mathbf{P}\mathbf{s}'}^2 (\Omega_{\mathbf{P}\mathbf{s}'}^2 - \varepsilon^2)^{-1}}{\frac{(2\pi)^d}{\lambda_R^{\mathbf{P}}} - \int d\mathbf{s} \chi_{\mathbf{P}\mathbf{s}}^2 \Omega_{\mathbf{P}\mathbf{s}}^4 (\Omega_{\mathbf{P}\mathbf{s}}^{-2} - (\Omega_{\mathbf{P}\mathbf{s}}^2 - \varepsilon^2)^{-1})},$$

$\varepsilon^2 \in \sigma(M_2)$ if and only if

$$\varepsilon = \omega_{\mathbf{P}/2-\mathbf{s}} + \omega_{\mathbf{P}/2+\mathbf{s}} \quad (75)$$

for some \mathbf{s} or

$$\frac{(2\pi)^d}{\lambda_R^{\mathbf{P}}} - \int d\mathbf{s} \chi_{\mathbf{P}\mathbf{s}}^2 \Omega_{\mathbf{P}\mathbf{s}}^4 (\Omega_{\mathbf{P}\mathbf{s}}^{-2} - (\Omega_{\mathbf{P}\mathbf{s}}^2 - \varepsilon^2)^{-1}) = 0. \quad (76)$$

Since $\omega_{\mathbf{P}/2+\mathbf{s}} + \omega_{\mathbf{P}/2-\mathbf{s}} \geq 2\omega_{\mathbf{P}/2} = \sqrt{\mathbf{P}^2 + 4\mu^2} > |\mathbf{P}|$, these values of ε obey the property $\varepsilon^2 \in (\mathbf{P}^2, \infty)$. It follows from eq.(64) that eq.(76) can be transformed to the form

$$\frac{1}{(2\pi)^{d+1}} \int_0^\infty d\alpha \int_0^\infty d\beta e^{-\mu^2(\alpha+\beta)} \left(\frac{\pi}{\alpha + \beta} \right)^{\frac{d+1}{2}} (e^{-\frac{\alpha\beta}{\alpha+\beta}(\mathbf{P}^2 - \varepsilon^2)} - 1) = -\frac{1}{\lambda_R^0}. \quad (77)$$

Since $\lambda_R^0 < 0$, the left-hand side of eq.(77) should be positive. This means $\varepsilon^2 > \mathbf{P}^2$. Lemma is proved.

4 Composed field operators

In the previous section we have constructed the Hamiltonian of the theory of "infinite number of fields" which was shown to be a self-adjoint operator in the Hilbert space. However, it is also necessary to check the property of the Poincare invariance.

To simplify the investigation, it is convenient to introduce an analog of the notion of a field which is very useful in traditional QFT: the Wightman axiomatic approach allows us to reduce the problem of Poincare invariance of the theory to the problem of Poincare invariance of Wightman functions.

However, it is not easy to introduce the field $\varphi^a(x)$ since we have not considered the nonsymmetric N -field states yet. However, one can investigate the properties of the "multifield operators":

$$\mathcal{W}_{N,k}(x_1, \dots, x_k) = \frac{1}{N} \sum_{a=1}^N \varphi^a(x_1) \dots \varphi^a(x_k). \quad (78)$$

$$\mathcal{W}_{N,k}(x_1, \dots, x_k) K_N f = K_N \tilde{\mathcal{W}}_{N,k}(x_1, \dots, x_k) f \quad (79)$$

with

$$\tilde{\mathcal{W}}_{N,k}(\mathbf{x}_1, \dots, \mathbf{x}_k) = \int D\phi |\Phi_0[\phi(\cdot)]|^2 \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_k) + N^{-1/2} \tilde{W}_k(\mathbf{x}_1, \dots, \mathbf{x}_k) + O(N^{-1}). \quad (80)$$

The operator \tilde{W}_k can be presented as

$$\begin{aligned} \tilde{W}_k(\mathbf{x}_1, \dots, \mathbf{x}_k) &= \int D\phi (\tilde{A}^+[\phi(\cdot)] + \tilde{A}^-[\phi(\cdot)]) \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_k) \Phi_0[\phi(\cdot)] = \\ &= \sum_{n=1}^{\infty} \int d\mathbf{p}_1 \dots d\mathbf{p}_n [A_{\mathbf{p}_1 \dots \mathbf{p}_n}^{(n)+}(\Phi_{\mathbf{p}_1 \dots \mathbf{p}_n}^{(n)}, \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_k) \Phi_0) + A_{\mathbf{p}_1 \dots \mathbf{p}_n}^{(n)-}(\Phi_{\mathbf{p}_1 \dots \mathbf{p}_n}^{(n)}, \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_k) \Phi_0)^*]. \end{aligned} \quad (81)$$

One can expect that Heisenberg operator will also obey the relation of the type (80):

$$\tilde{\mathcal{W}}_{N,k}(x_1, \dots, x_k) = (\Phi_0[\phi(\cdot)], \phi(x_1) \dots \phi(x_k) \Phi_0[\phi(\cdot)]) + N^{-1/2} \tilde{W}_k(x_1, \dots, x_k) + O(N^{-1}). \quad (82)$$

Here $\phi(x)$ is a Heisenberg operator of the free field of the mass μ . The property (82) is to be checked in appendix B.

The multifield operators $\tilde{W}_k(x_1, \dots, x_k)$ being analogs of fields are to be investigated.

4.1 Multifield operators

In this subsection we compute the explicit form of the operators $\tilde{W}_k(x_1, \dots, x_k)$.

The " k -field" (78) satisfies the Heisenberg equation

$$\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + m^2 + \frac{\lambda}{N} \sum_{b=1}^N \varphi^b(x_A) \varphi^b(x_A) \right) \mathcal{W}_{N,k}(x_1, \dots, x_k) = 0.$$

The property (79) implies that

$$\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + m^2 + \lambda \tilde{\mathcal{W}}_{N,2}(x_A, x_A) \right) \tilde{\mathcal{W}}_{N,k}(x_1, \dots, x_k) = 0.$$

Use now the property (82). One can notice that $m^2 + \lambda(\Phi_0, \phi(x_A) \phi(x_A) \Phi_0) = \mu^2$. Therefore,

$$\begin{aligned} &\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + \mu^2 + \frac{\lambda}{\sqrt{N}} \tilde{W}_2(x_A, x_A) + O(N^{-1}) \right) \\ &((\Phi_0, \phi(x_1) \dots \phi(x_k) \Phi_0) + N^{-1/2} \tilde{W}_k(x_1, \dots, x_k) + O(N^{-1})) = 0. \end{aligned} \quad (83)$$

The terms of order $O(1)$ give us an equation on the vacuum average value. It has the form

$$\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + \mu^2 \right) (\Phi_0, \phi(x_1) \dots \phi(x_k) \Phi_0) = 0.$$

This equation is automatically satisfied. The terms of order $O(N^{1/2})$ lead to the nontrivial equation:

$$\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + \mu^2 \right) \tilde{W}_k(x_1, \dots, x_k) + Q(x_A) (\Phi_0, \phi(x_1) \dots \phi(x_k) \Phi_0) = 0. \quad (84)$$

with

$$Q(x_A) = \lambda \tilde{W}_2(x_A, x_A). \quad (85)$$

fields W_k . It follows from the Wick operators (81) can be presented as

$$\tilde{W}_k(\mathbf{x}_1, \dots, \mathbf{x}_k) = \sum (\Phi_0, \phi(\mathbf{x}_{l_1^1}) \phi(\mathbf{x}_{l_1^2}) \Phi_0) \dots (\Phi_0, \phi(\mathbf{x}_{l_\nu^1}) \phi(\mathbf{x}_{l_\nu^2}) \Phi_0) \hat{W}_{k-2\nu}(\mathbf{x}_{m_1}, \dots, \mathbf{x}_{m_{k-2\nu}}). \quad (86)$$

Here the summation is performed over all decompositions of the set

$$\{1, 2, \dots, k\} = \{l_1^1, l_1^2\} \cup \dots \cup \{l_\nu^1, l_\nu^2\} \cup \{m_1, \dots, m_{k-2\nu}\},$$

while

$$l_1^1 < l_1^2, \quad l_\nu^1 < l_\nu^2, \quad m_1 < \dots < m_{k-2\nu}, \quad k - 2\nu > 0.$$

The operator $\hat{W}_s(\mathbf{x}_1, \dots, \mathbf{x}_s)$ entering to the formula (86) has the form

$$\begin{aligned} \hat{W}_s(\mathbf{x}_1, \dots, \mathbf{x}_s) = \int d\mathbf{p}_1 \dots d\mathbf{p}_s [A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)+}(\Phi_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)}, : \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_s) : \Phi_0) \\ + A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)-}(\Phi_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)}, : \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_s) : \Phi_0)^*]. \end{aligned} \quad (87)$$

The notation $::$ is used for the Wick ordering of combinations of fields

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int \frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} [a_{\mathbf{k}}^+ e^{-i\mathbf{k}\mathbf{x}} + a_{\mathbf{k}}^- e^{i\mathbf{k}\mathbf{x}}]$$

Analogously, define the Heisenberg operators $\hat{W}_s(x_1, \dots, x_s)$ from the recursive relations

$$\tilde{W}_k(x_1, \dots, x_k) = \sum (\Phi_0, \phi(x_{l_1^1}) \phi(x_{l_1^2}) \Phi_0) \dots (\Phi_0, \phi(x_{l_\nu^1}) \phi(x_{l_\nu^2}) \Phi_0) \hat{W}_{k-2\nu}(x_{m_1}, \dots, x_{m_{k-2\nu}}). \quad (88)$$

Applying the Wick theorem to the combinations of the field and momenta operators

$$\pi(\mathbf{x}) = \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})} = \frac{1}{(2\pi)^{d/2}} \int \frac{d\mathbf{k}}{\sqrt{2}} i \sqrt{\omega_{\mathbf{k}}} [a_{\mathbf{k}}^+ e^{-i\mathbf{k}\mathbf{x}} + a_{\mathbf{k}}^- e^{i\mathbf{k}\mathbf{x}}]$$

we obtain in an analogous way that

$$\begin{aligned} \frac{\partial}{\partial x_{i_1}^0} \dots \frac{\partial}{\partial x_{i_l}^0} \hat{W}_s(x_1, \dots, x_s) = \int d\mathbf{p}_1 \dots d\mathbf{p}_s [A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)+}(\Phi_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)}, : \phi(\mathbf{x}_1) \dots \pi_{i_1}(\mathbf{x}_{i_1}) \dots \pi_{i_l}(\mathbf{x}_{i_l}) \dots \phi(\mathbf{x}_s) : \Phi_0) \\ + A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)-}(\Phi_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)}, : \phi(\mathbf{x}_1) \dots \pi_{i_1}(\mathbf{x}_{i_1}) \dots \pi_{i_l}(\mathbf{x}_{i_l}) \phi(\mathbf{x}_s) : \Phi_0)^*] \end{aligned} \quad (89)$$

for $i_1 < \dots < i_l$, $x_1^0 = \dots = x_s^0 = 0$.

Let us find an equation on the operator \hat{W}_s . For $k = 2$, $\tilde{W}_2 = \hat{W}_2$, so that

$$\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + \mu^2 \right) \hat{W}_2(x_1, x_2) + Q(x_A) (\Phi_0, \phi(x_1) \phi(x_2) \Phi_0) = 0. \quad (90)$$

For odd values of k , one has $(\Phi_0, \phi(x_1) \dots \phi(x_k) \Phi_0) = 0$, so that

$$\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + \mu^2 \right) \tilde{W}_k(x_1, \dots, x_k) = 0.$$

It follows from the recursive relations that

$$\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + \mu^2 \right) \hat{W}_k(x_1, \dots, x_k) = 0. \quad (91)$$

$A = 1$. The general case can be investigated analogously. The quantity

$$\left(\frac{\partial}{\partial x_1^\mu} \frac{\partial}{\partial x_{1\mu}} + \mu^2 \right) \tilde{W}_k(x_1, \dots, x_k)$$

entering to the left-hand side of eq.(84) can be decomposed into two parts. One of them corresponds to the case $k - 2\nu = 2$, another - to $k - 2\mu > 2$. The first part is

$$\begin{aligned} \left(\frac{\partial}{\partial x_1^\mu} \frac{\partial}{\partial x_{1\mu}} + \mu^2 \right) \sum (\Phi_0, \phi(x_{l_1^1}) \phi(x_{l_1^2}) \Phi_0) \dots (\Phi_0, \phi(x_{l_\nu^1}) \phi(x_{l_\nu^2}) \Phi_0) \hat{W}_2(x_1, x_{m_2}) = \\ -Q(x_1) \sum (\Phi_0, \phi(x_1) \phi(x_{m_2}) \Phi_0) (\Phi_0, \phi(x_{l_1^1}) \phi(x_{l_1^2}) \Phi_0) \dots (\Phi_0, \phi(x_{l_\nu^1}) \phi(x_{l_\nu^2}) \Phi_0) \\ = -Q(x_1) (\Phi_0, \phi(x_1) \dots \phi(x_k) \Phi_0). \end{aligned} \quad (92)$$

The second part reads

$$\left(\frac{\partial}{\partial x_1^\mu} \frac{\partial}{\partial x_{1\mu}} + \mu^2 \right) \sum_{k-2\nu>2} (\Phi_0, \phi(x_{l_1^1}) \phi(x_{l_1^2}) \Phi_0) \dots (\Phi_0, \phi(x_{l_\nu^1}) \phi(x_{l_\nu^2}) \Phi_0) \hat{W}_{k-2\nu}(x_{m_1}, \dots, x_{m_{k-2\nu}}). \quad (93)$$

It follows from eq.(84) then that the quantity (93) should vanish. We obtain by induction that the functions \hat{W}_4, \hat{W}_6 etc. obey eq.(91).

To find an explicit form of \hat{W}_k , prove the following proposition.

Proposition 4.1. *Let*

$$\left(\frac{\partial}{\partial x_A^\mu} \frac{\partial}{\partial x_{A\mu}} + \mu^2 \right) f_k(x_1, \dots, x_k) = 0, \quad A = \overline{1, k} \quad (94)$$

and

$$\frac{\partial}{\partial x_{i_1}^\mu} \dots \frac{\partial}{\partial x_{i_l}^\mu} f_k(x_1, \dots, x_k) = 0, \quad i_1 < \dots < i_l, \quad x_1^0 = \dots = x_k^0 = 0$$

Then $f_k = 0$.

Proof. Consider the spatial Fourier transformation $\tilde{f}_k(\mathbf{p}_1, t_1; \dots; \mathbf{p}_k, t_k)$ of the function f_k . It obeys the set of equations

$$\left(\frac{\partial}{\partial t_A} \frac{\partial}{\partial t_A} + \omega_{\mathbf{p}_A^2} \right) \tilde{f}_k(\mathbf{p}_1, t_1; \dots; \mathbf{p}_k, t_k) = 0, \quad A = \overline{1, k}$$

with $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + \mu^2}$. This implies that

$$\tilde{f}_k(\mathbf{p}_1, t_1; \dots; \mathbf{p}_k, t_k) = \sum_{\sigma_1, \dots, \sigma_k \in \{-1, 1\}} \alpha_{\sigma_1 \dots \sigma_k}(\mathbf{p}_1, \dots, \mathbf{p}_k) e^{i\sigma_1 \omega_{\mathbf{p}_1} t_1 + \dots + i\sigma_k \omega_{\mathbf{p}_k} t_k}.$$

One can express the coefficients α as

$$\alpha_{\sigma_1 \dots \sigma_k}(\mathbf{p}_1, \dots, \mathbf{p}_k) = \left(\frac{1}{2} - \frac{i\sigma_1}{2\omega_{\mathbf{p}_1}} \right) \dots \left(\frac{1}{2} - \frac{i\sigma_k}{2\omega_{\mathbf{p}_k}} \right) \tilde{f}_k(\mathbf{p}_1, t_1, \dots, \mathbf{p}_k, t_k)$$

Therefore, $\alpha_{\sigma_1 \dots \sigma_k}(\mathbf{p}_1, \dots, \mathbf{p}_k) = 0$. This implies $f_k = 0$. Proposition is proved.

Denote

$$\begin{aligned} \hat{W}_s^0(x_1, \dots, x_s) = \int d\mathbf{p}_1 \dots d\mathbf{p}_s [A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)+}(\Phi_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)}, : \phi(x_1) \dots \phi(x_s) : \Phi_0) \\ + A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)-}(\Phi_{\mathbf{p}_1 \dots \mathbf{p}_s}^{(s)}, : \phi(x_1) \dots \phi(x_s) : \Phi_0)^*]. \end{aligned} \quad (95)$$

$f_k = 0$, so that $W_s = W_s^0$. The explicit form of the operator W_s is

$$\begin{aligned} \hat{W}_s(\mathbf{x}_1, t_1, \dots, \mathbf{x}_s, t_s) = & \frac{1}{(2\pi)^{sd/2}} \int \frac{d\mathbf{p}_1}{\sqrt{2\omega_{\mathbf{p}_1}}} \dots \frac{d\mathbf{p}_s}{\sqrt{2\omega_{\mathbf{p}_s}}} \sqrt{k!} (A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{+(s)} e^{i\omega_{\mathbf{p}_1} t_1 + \dots + i\omega_{\mathbf{p}_s} t_s - i\mathbf{p}_1 \mathbf{x}_1 - \dots - i\mathbf{p}_s \mathbf{x}_s} \\ & + A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{-(s)} e^{-i\omega_{\mathbf{p}_1} t_1 - \dots - i\omega_{\mathbf{p}_s} t_s + i\mathbf{p}_1 \mathbf{x}_1 + \dots + i\mathbf{p}_s \mathbf{x}_s}) \end{aligned} \quad (96)$$

Let $f \in \mathcal{S}(\mathbf{R}^{ds})$. Consider the operators

$$\hat{W}_s[f] = \int dx_1 \dots dx_s \hat{W}_s(x_1, \dots, x_s) f(x_1, \dots, x_s).$$

They are defined on the set \check{D} of all finite vectors of $\check{\mathcal{F}}$. The set \check{D} is invariant under the operator $\hat{W}_s[f]$.

Proposition 4.2. 1. $\hat{W}_s(x_1, \dots, x_s)$ ($s \neq 2$) is an operator distribution. 2. The set

$$\hat{W}_{s_1}[f_1] \dots \hat{W}_{s_k}[f_k] |0\rangle, \quad f_j \in \mathcal{S}(\mathbf{R}^{dj})$$

is a total set in $\check{\mathcal{F}}$. The first statement is obvious. The second statement is a corollary of lemma A.14.

Thus, we see that the "k-field" \tilde{W}_k has the form (88), the operators \hat{W}_s having the form (95) are already found. The remaining problem is to find the explicit form of the bifield $\hat{W}_2(x_1, x_2)$. Since an equation for the bifield contains the operator $Q(x)$ being an analog of the composed $\lambda\varphi^a\varphi^a$ field, let us investigate first its properties.

4.2 The $\lambda\varphi^a\varphi^a$ composed field

The operator $Q(x) = Q(\mathbf{x}, t)$ can be presented as

$$\begin{aligned} Q(\mathbf{x}, t) = & e^{iHt} \lambda \hat{W}_2(\mathbf{x}, 0, \mathbf{x}, 0) e^{-iHt} = \\ & \lambda \frac{\sqrt{2}}{(2\pi)^d} \int \frac{d\mathbf{k}_1}{\sqrt{2\omega_{\mathbf{k}_1}}} \frac{d\mathbf{k}_2}{\sqrt{2\omega_{\mathbf{k}_2}}} (A_{\mathbf{k}_1 \mathbf{k}_2}^+(t) e^{-i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}} + A_{\mathbf{k}_1 \mathbf{k}_2}^-(t) e^{i(\mathbf{k}_1 + \mathbf{k}_2)\mathbf{x}}), \end{aligned} \quad (97)$$

where

$$A_{\mathbf{k}_1 \mathbf{k}_2}^\pm(t) = e^{iHt} A_{\mathbf{k}_1 \mathbf{k}_2}^{\pm(2)} e^{-iHt}.$$

After transformations (51) and (55) expression (97) takes the form

$$\begin{aligned} Q(\mathbf{x}, t) = & \lambda \frac{\sqrt{2}}{(2\pi)^d} \int d\mathbf{P} d\mathbf{s} \sqrt{\frac{2\Omega_{\mathbf{P}\mathbf{s}}}{2\omega_{\mathbf{P}/2+\mathbf{s}} 2\omega_{\mathbf{P}/2-\mathbf{s}}}} e^{-i\mathbf{P}\mathbf{x}} Q_{\mathbf{P}\mathbf{s}}(t) = \\ = & \int d\mathbf{P} d\mathbf{s} (C_{\mathbf{P}\mathbf{s}}^+ \gamma_{\mathbf{P}\mathbf{s}}(\mathbf{x}, t) + C_{\mathbf{P}\mathbf{s}}^- \gamma_{\mathbf{P}\mathbf{s}}^*(\mathbf{x}, t)) = C^+[\gamma(\mathbf{x}, t)] + C^-[\gamma(\mathbf{x}, t)], \end{aligned}$$

where

$$\gamma(\mathbf{x}, t) = \frac{\sqrt{2}}{(2\pi)^d} e^{-i\mathbf{P}\mathbf{x}} \frac{1}{\sqrt{2M}} e^{iMt} \lambda \Omega \chi. \quad (98)$$

Our purpose is to prove that $Q(x)$ is an operator distribution. Therefore, we should show that $(\chi(\mathbf{x}, t))_{\mathbf{P}\mathbf{s}}$ can be viewed as a vector distribution. However, an infinite quantity λ^{-1} and function $\Omega^2 \chi \notin L^2$ enter to eq.(98). It is remarkable that these divergences can be eliminated: one can use the property:

$$M^{-2} \lambda \Omega^2 \chi = \lambda_R \chi.$$

Here λ_R is an operator of multiplication by $\lambda_R^{\mathbf{P}}$. Thus, the vector function (98) can be written as

$$\gamma(\mathbf{x}, t) = (2\pi)^{-d} e^{-i\mathbf{P}\mathbf{x}} e^{iMt} M^{3/2} \lambda_R \chi. \quad (99)$$

$$\gamma(\mathbf{x}, t) = (2\pi)^{-d}(-\Delta + 1)^m \left(-\frac{\partial^2}{\partial t^2} \right) e^{-i\mathbf{P}\mathbf{x}} e^{iMt} M^{-1/2} (\mathbf{P}^2 + 1)^{-m} \lambda_R \chi.$$

Since $M^{-1/2}(\mathbf{P}^2 + 1)^{-m} \lambda_R \chi \in \mathcal{H}_2$ for sufficiently large m , the function $e^{-i\mathbf{P}\mathbf{x}} e^{iMt} M^{-1/2} (\mathbf{P}^2 + 1)^{-m} \lambda_R \chi$ is a bounded continuous vector function, we obtain from lemmas A.12 and A.13 that γ is a vector distribution. Lemma A.14 implies that $Q(\mathbf{x}, t)$ is an operator distribution.

4.3 Canonical variables as operator distributions

The purpose of this subsection is to investigate the properties of the operators $Q_{\mathbf{P}\mathbf{s}}$ and $\Pi_{\mathbf{P}\mathbf{s}}$. These properties will be essentially used.

First of all, notice that

$$Q_{\mathbf{P}\mathbf{s}} = C^+[\xi_{\mathbf{P}\mathbf{s}}] + C^-[\xi_{-\mathbf{P}\mathbf{s}}], \quad \Pi_{\mathbf{P}\mathbf{s}} = C^+[\pi_{\mathbf{P}\mathbf{s}}] + C^-[\pi_{-\mathbf{P}\mathbf{s}}],$$

where $\xi_{\mathbf{P}\mathbf{s}}$ and $\pi_{\mathbf{P}\mathbf{s}}$ have the form

$$(\xi_{\mathbf{P}\mathbf{s}})_{\mathbf{P}'\mathbf{s}'} = \delta_{\mathbf{P}\mathbf{P}'} ((2M_{\mathbf{P}}^{-1/2})_{\mathbf{s}\mathbf{s}'}), \quad (\pi_{\mathbf{P}\mathbf{s}})_{\mathbf{P}'\mathbf{s}'} = i\delta_{\mathbf{P}\mathbf{P}'} ((M_{\mathbf{P}}/2)^{-1/2})_{\mathbf{s}\mathbf{s}'},$$

Consider the integrals

$$\int d\mathbf{P} d\mathbf{s} \varphi_{\mathbf{P}\mathbf{s}} \xi_{\mathbf{P}\mathbf{s}} = (2M)^{-1/2} \bar{\varphi}, \quad (100)$$

$$\int d\mathbf{P} d\mathbf{s} \varphi_{\mathbf{P}\mathbf{s}} \pi_{\mathbf{P}\mathbf{s}} = i(M/2)^{1/2} \bar{\varphi} \quad (101)$$

where $\varphi_{\mathbf{P}\mathbf{s}}$ are complex functions from $\mathcal{S}(\mathbf{R}^{2d})$,

$$\bar{\varphi}_{\mathbf{P}\mathbf{s}} = \frac{1}{2}(\varphi_{\mathbf{P}\mathbf{s}} + \varphi_{\mathbf{P}, -\mathbf{s}}) \quad (102)$$

Since $(2M)^{-1/2}$ is a bounded operator, the integral (100) is always defined. However, $(M/2)^{1/2}$ is not a bounded operator, so that quantity (101) may be not defined. We see that the expression for $\xi_{\mathbf{P}\mathbf{s}}$ defines a vector distribution, while $\pi_{\mathbf{P}\mathbf{s}}$ is not a vector distribution.

To consider objects like $\pi_{\mathbf{P}\mathbf{s}}$ as vector distributions, it is necessary to perform the renormalization procedure. Let

$$Q_{\mathbf{P}}(t) = \frac{1}{(2\pi)^d} \int d\mathbf{x} e^{i\mathbf{P}\mathbf{x}} Q(\mathbf{x}, t) = -\ddot{R}_{\mathbf{P}}(t),$$

$$R_{\mathbf{P}}(t) = \frac{1}{(2\pi)^d} \int d\mathbf{x} e^{i\mathbf{P}\mathbf{x}} R(\mathbf{x}, t) = C^+[r_{\mathbf{P}}(t)] + C^-[r_{-\mathbf{P}}(t)],$$

where

$$(r_{\mathbf{P}}(t))_{\mathbf{P}'\mathbf{s}'} = (2\pi)^{-d} \delta_{\mathbf{P}\mathbf{P}'} (e^{iMt} M^{-1/2} \lambda_R \chi)_{\mathbf{P}'\mathbf{s}'}$$

is a vector distribution.

Denote

$$\pi_{\mathbf{P}\mathbf{s}}^{ren} = \pi_{\mathbf{P}\mathbf{s}} - \frac{1}{\sqrt{2}} \Omega_{\mathbf{P}\mathbf{s}}^2 \chi_{\mathbf{P}\mathbf{s}} \dot{r}_{\mathbf{P}}(0), \quad (103)$$

where $\chi_{\mathbf{P}\mathbf{s}}$ has the form (61). One has

$$\int d\mathbf{P} d\mathbf{s} \pi_{\mathbf{P}\mathbf{s}}^{ren} \varphi_{\mathbf{P}\mathbf{s}} = i(M/2)^{1/2} [\bar{\varphi} - \frac{\lambda^R}{(2\pi)^d} \chi \int d\mathbf{s}' \bar{\varphi}_{\mathbf{P}\mathbf{s}'} \Omega_{\mathbf{P}\mathbf{s}'}^2 \chi_{\mathbf{P}\mathbf{s}'}] = i(M/2)^{1/2} M^{-2} \Omega^2 \bar{\varphi}.$$

A.14 the following proposition.

Proposition 4.5. $Q_{\mathbf{P}\mathbf{s}}$ and

$$\Pi_{\mathbf{P}\mathbf{s}}^{ren} = \Pi_{\mathbf{P}\mathbf{s}} - \frac{1}{\sqrt{2}} \Omega_{\mathbf{P}\mathbf{s}}^2 \chi_{\mathbf{P}\mathbf{s}} \dot{R}_{\mathbf{P}}(0) \quad (104)$$

are operator distributions.

Investigate now the transformation properties of these distributions. Analogously to the previous subsection, we obtain

Proposition 4.6. *For spatial rotations, the following properties are satisfied:*

$$u_{\Lambda,0} \xi_{\mathbf{P},\mathbf{s}} = \xi_{\Lambda\mathbf{P},\Lambda\mathbf{s}}, \quad u_{\Lambda,0} \pi_{\mathbf{P},\mathbf{s}}^{ren} = \pi_{\Lambda\mathbf{P},\Lambda\mathbf{s}}^{ren}, \quad u_{\Lambda,0} r_{\mathbf{P}}(t) = r_{\Lambda\mathbf{P}}(t).$$

Corollary. *Under conditions (67) the operators $Q_{\mathbf{P}\mathbf{s}}$, $\Pi_{\mathbf{P}\mathbf{s}}^{ren}$ obey the following transformation properties*

$$\begin{aligned} U_{\Lambda,0} Q_{\mathbf{P}\mathbf{s}} U_{\Lambda,0}^{-1} &= Q_{\Lambda\mathbf{P},\Lambda\mathbf{s}}, & U_{\Lambda,0} \Pi_{\mathbf{P}\mathbf{s}}^{ren} U_{\Lambda,0}^{-1} &= \Pi_{\Lambda\mathbf{P},\Lambda\mathbf{s}}^{ren}, & U_{\Lambda,0} R_{\mathbf{P}}(t) U_{\Lambda,0}^{-1} &= R_{\Lambda\mathbf{P}}(t), \\ U_{1,\mathbf{a}} Q_{\mathbf{P}\mathbf{s}} U_{1,\mathbf{a}}^{-1} &= e^{-i\mathbf{P}\mathbf{a}} Q_{\mathbf{P},\mathbf{s}}, & U_{1,\mathbf{a}} \Pi_{\mathbf{P}\mathbf{s}}^{ren} U_{1,\mathbf{a}}^{-1} &= e^{-i\mathbf{P}\mathbf{a}} \Pi_{\mathbf{P},\mathbf{s}}^{ren}, & U_{1,\mathbf{a}} R_{\mathbf{P}}(t) U_{1,\mathbf{a}}^{-1} &= R_{\mathbf{P}}(t + a_0). \end{aligned} \quad (105)$$

4.4 The bifield operator

In this section we construct the bifield operator $\hat{W}_2(x_1, x_2)$ which obey eq.(90) and initial conditions (89). We show it to be an operator distribution of $\mathbf{x}_1, \mathbf{x}_2$ at fixed values of x_1^0, x_2^0 . It can be also viewed as an operator distribution of x_1, x_2 .

Firs of all, consider the spatial Fourier transformation

$$\hat{W}_2(\mathbf{x}, t_x; \mathbf{y}, t_y) = \frac{1}{(2\pi)^d} \int d\mathbf{k} d\mathbf{p} w_2(\mathbf{k}, t_x; \mathbf{p}, t_y) e^{-i\mathbf{k}\mathbf{x} - i\mathbf{p}\mathbf{y}}.$$

Initial conditions (89) can be presented in the following form

$$\begin{aligned} w_2(\mathbf{k}, 0; \mathbf{p}, 0) &= \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}} \omega_{\mathbf{p}}}} Q_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}}, \\ \frac{\partial}{\partial t_x} w_2(\mathbf{k}, 0; \mathbf{p}, 0) &= \sqrt{\frac{\omega_{\mathbf{k}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}{\omega_{\mathbf{p}}}} \Pi_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}}, \\ \frac{\partial}{\partial t_y} w_2(\mathbf{k}, 0; \mathbf{p}, 0) &= \sqrt{\frac{\omega_{\mathbf{p}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}{\omega_{\mathbf{k}}}} \Pi_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}}, \\ \frac{\partial}{\partial t_x} \frac{\partial}{\partial t_y} w_2(\mathbf{k}, 0; \mathbf{p}, 0) &= \sqrt{(\omega_{\mathbf{p}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})) \omega_{\mathbf{k}}} Q_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}}. \end{aligned} \quad (106)$$

Eqs.(90) can be written as

$$\begin{aligned} \left(\frac{\partial^2}{\partial t_x \partial t_x} + \omega_{\mathbf{k}}^2 \right) w_2(\mathbf{k}, t_x; \mathbf{p}, t_y) + \frac{1}{2\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}}(t_x - t_y)} Q_{\mathbf{k}+\mathbf{p}}(t_x) &= 0, \\ \left(\frac{\partial^2}{\partial t_y \partial t_y} + \omega_{\mathbf{p}}^2 \right) w_2(\mathbf{k}, t_x; \mathbf{p}, t_y) + \frac{1}{2\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}(t_x - t_y)} Q_{\mathbf{k}+\mathbf{p}}(t_y) &= 0. \end{aligned} \quad (107)$$

$$\begin{aligned}
w_2(\mathbf{k}, t_x; \mathbf{p}, t_y) = & \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}} Q_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}} \cos(\omega_{\mathbf{k}}t_x + \omega_{\mathbf{p}}t_y) \\
& + \frac{1}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{p}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}} \Pi_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}} \sin(\omega_{\mathbf{k}}t_x + \omega_{\mathbf{p}}t_y) \\
& - \int_0^{t_x} d\tau \frac{\sin(\omega_{\mathbf{k}}(t_x - \tau))}{\omega_{\mathbf{k}}} \frac{1}{2\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}}(\tau - t_y)} Q_{\mathbf{k}+\mathbf{p}}(\tau) \\
& - \int_0^{t_y} d\tau \frac{\sin(\omega_{\mathbf{p}}(t_y - \tau))}{\omega_{\mathbf{p}}} \frac{1}{2\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}(t_x - \tau)} Q_{\mathbf{k}+\mathbf{p}}(\tau).
\end{aligned} \tag{108}$$

This form is not suitable for investigation since $\Pi_{\mathbf{P}\mathbf{s}}$ has been discovered to be not a distribution, while $Q_{\mathbf{P}\mathbf{s}}$ is a distribution rather than ordinary function. However, we can use the relation $Q = -\ddot{R}$ and integrate by parts. We obtain:

$$\begin{aligned}
w_2(\mathbf{k}, t_x; \mathbf{p}, t_y) = & \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}} Q_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}} \cos(\omega_{\mathbf{k}}t_x + \omega_{\mathbf{p}}t_y) \\
& + \frac{1}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{p}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}} \Pi_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}}^{\text{ren}} \sin(\omega_{\mathbf{k}}t_x + \omega_{\mathbf{p}}t_y) \\
& - \int_0^{t_x} d\tau \frac{\partial}{\partial \tau} \left(\frac{\sin(\omega_{\mathbf{k}}(t_x - \tau))}{\omega_{\mathbf{k}}} \frac{1}{2\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}}(\tau - t_y)} \right) \frac{\partial}{\partial \tau} R_{\mathbf{k}+\mathbf{p}}(\tau) \\
& - \int_0^{t_y} d\tau \frac{\partial}{\partial \tau} \left(\frac{\sin(\omega_{\mathbf{p}}(t_y - \tau))}{\omega_{\mathbf{p}}} \frac{1}{2\omega_{\mathbf{k}}} e^{-i\omega_{\mathbf{k}}(t_x - \tau)} \right) \frac{\partial}{\partial \tau} R_{\mathbf{k}+\mathbf{p}}(\tau).
\end{aligned} \tag{109}$$

Since for any smooth function φ the integral

$$\int_0^t d\tau \dot{R}_{\mathbf{P}}(\tau) \varphi(\tau) = R_{\mathbf{P}}(\tau) \varphi(\tau) \Big|_0^t - \int_0^t d\tau \dot{\varphi}(\tau) R_{\mathbf{P}}(\tau)$$

is defined as an operator distribution of \mathbf{P} ($R_{\mathbf{P}}(\tau)$ is a distribution of \mathbf{P} at fixed τ), while $Q_{\mathbf{P}\mathbf{s}}$ and $\Pi_{\mathbf{P}\mathbf{s}}^{\text{ren}}$ are operator distributions, expression (109) gives us an operator distribution. Thus, we obtain the following proposition.

Proposition 4.7. $w_2(\mathbf{k}, t_x; \mathbf{p}, t_y)$ is:

- (1) an operator distribution of \mathbf{k}, \mathbf{p} at fixed t_x, t_y ;
- (2) an operator distribution of $\mathbf{k}, t_x, \mathbf{p}, t_y$.

Corollary of proposition 4.6 implies the following statement.

Proposition 4.8. The transformation properties of w_2 under spatial rotations and translations are:

$$\begin{aligned}
U_{\Lambda,0} w_2(\mathbf{k}, t_x, \mathbf{p}, t_y) U_{\Lambda,0}^{-1} &= w_2(\Lambda \mathbf{k}, t_x, \Lambda \mathbf{p}, t_y) \\
U_{1,a} w_2(\mathbf{k}, t_x, \mathbf{p}, t_y) U_{1,a}^{-1} &= e^{-i(\mathbf{k}+\mathbf{p})\mathbf{a}} w_2(\mathbf{k}, t_x; \mathbf{p}, t_y).
\end{aligned} \tag{110}$$

Corollary. $\hat{W}(\mathbf{x}, t_x; \mathbf{y}, t_y)$ is an t_x, t_y -dependent operator distribution of \mathbf{x}, \mathbf{y} with the following transformation properties under spatial rotations and translations:

$$\begin{aligned}
U_{\Lambda,0} \hat{W}_2(\mathbf{x}, t_x, \mathbf{y}, t_y) U_{\Lambda,0}^{-1} &= \hat{W}_2(\Lambda \mathbf{x}, t_x, \Lambda \mathbf{y}, t_y) \\
U_{1,a} \hat{W}_2(\mathbf{x}, t_x, \mathbf{y}, t_y) U_{1,a}^{-1} &= \hat{W}_2(\mathbf{x} + \mathbf{a}, t_x; \mathbf{y} + \mathbf{a}, t_y).
\end{aligned} \tag{111}$$

$\hat{W}(x, y)$ is also an operator distribution of x, y .

$$W_2[f, t_x, t_y] = \int d\mathbf{x}d\mathbf{y} \hat{W}_2(\mathbf{x}, t_x; \mathbf{y}, t_y) f(\mathbf{x}, \mathbf{y}).$$

Proposition 4.9. *The set of all finite linear combinations*

$$\sum_n W_2[f_{n,1}, t_{x,1}^n, t_{y,1}^n] \dots W_2[f_{n,s_n}, t_{x,s_n}^n, t_{y,s_n}^n] |0\rangle \quad (112)$$

is dense in $\mathcal{F}(\mathcal{H}_2)$.

To prove this proposition, it is sufficient to consider the case $t_{x,i}^n = t_{y,i}^n = 0$ only and use lemma A.15. One can also consider the operators

$$W_2[g] = \int dt_x dt_y d\mathbf{x}d\mathbf{y} W_2(\mathbf{x}, t_x; \mathbf{y}, t_y) g(\mathbf{x}, t_x, \mathbf{y}, t_y).$$

Proposition 4.10. *The set of all finite linear combinations*

$$\sum_n W_2[g_{n,1}] \dots W_2[g_{n,s_n}] |0\rangle \quad (113)$$

is dense in $\mathcal{F}(\mathcal{H}_2)$.

To prove this proposition, it is sufficient to approximate the vector (112) by the vector (113) by choosing

$$g_{n,k}(\mathbf{x}, t_x; \mathbf{y}, t_y) = f_{n,k}(\mathbf{x}, \mathbf{y}) \frac{1}{\varepsilon^2} \varphi(t_x/\varepsilon, t_y/\varepsilon)$$

for any smooth function $\varphi(\tau_x, \tau_y)$ with compact support such that $\int d\tau_x d\tau_y \varphi(\tau_x, \tau_y) = 1$.

Thus, the cyclic property of the vacuum state is checked.

4.5 Invariance under time translations

The purpose of this subsection is to check the invariance property of the bifield under time translations:

$$U_{1,a} \hat{W}_2(\mathbf{x}, t_x, \mathbf{y}, t_y) U_{1,a}^{-1} = \hat{W}_2(\mathbf{x}, t_x + t, \mathbf{y}, t_y + t) \quad (114)$$

if $a^0 = t$, $\mathbf{a} = 0$. Let us prove first the following proposition. Denote

$$Q_{\mathbf{P}\mathbf{s}}(T) = e^{iHT} Q_{\mathbf{P}\mathbf{s}} e^{-iHT}, \quad \Pi_{\mathbf{P}\mathbf{s}}^{ren}(T) = e^{iHT} \Pi_{\mathbf{P}\mathbf{s}}^{ren} e^{-iHT}.$$

For smooth function $f(\tau)$ let

$$\int_0^T d\tau f(\tau) \frac{\partial}{\partial \tau} R_{\mathbf{P}}(\tau) \equiv f(\tau) R_{\mathbf{P}}(\tau) \Big|_0^T - \int_0^T d\tau \frac{\partial f}{\partial \tau} R_{\mathbf{P}}(\tau).$$

Proposition 4.11. *The following properties are satisfied:*

$$\begin{aligned} Q_{\mathbf{P}\mathbf{s}}(T) &= Q_{\mathbf{P}\mathbf{s}} \cos(\Omega_{\mathbf{P}\mathbf{s}} T) + \Pi_{\mathbf{P}\mathbf{s}}^{ren} \frac{\sin(\Omega_{\mathbf{P}\mathbf{s}} T)}{\Omega_{\mathbf{P}\mathbf{s}}} - \\ &\int_0^T \frac{d\tau}{2\sqrt{\omega_{\mathbf{P}/2-\mathbf{s}} \omega_{\mathbf{P}/2+\mathbf{s}} \Omega_{\mathbf{P}\mathbf{s}}}} \frac{\partial}{\partial \tau} [\sin(\Omega_{\mathbf{P}\mathbf{s}}(T-\tau))] \frac{\partial}{\partial \tau} R_{\mathbf{P}}(\tau), \\ \Pi_{\mathbf{P}\mathbf{s}}^{ren}(T) &= \Pi_{\mathbf{P}\mathbf{s}}^{ren} \cos(\Omega_{\mathbf{P}\mathbf{s}} T) - Q_{\mathbf{P}\mathbf{s}} \Omega_{\mathbf{P}\mathbf{s}} \sin(\Omega_{\mathbf{P}\mathbf{s}} T) - \\ &\int_0^T \frac{d\tau}{2} \sqrt{\frac{\Omega_{\mathbf{P}\mathbf{s}}}{\omega_{\mathbf{P}/2-\mathbf{s}} \omega_{\mathbf{P}/2+\mathbf{s}}}} \frac{\partial}{\partial \tau} [\cos(\Omega_{\mathbf{P}\mathbf{s}}(T-\tau))] \frac{\partial}{\partial \tau} R_{\mathbf{P}}(\tau) \end{aligned} \quad (115)$$

Proof. First of all, notice that

$$\begin{aligned} Q_{\mathbf{P}\mathbf{s}}(T) &= C^+[\xi_{\mathbf{P}\mathbf{s}}(T)] + C^-[\xi_{-\mathbf{P}\mathbf{s}}(T)], \\ \Pi_{\mathbf{P}\mathbf{s}}^{ren}(T) &= C^+[\pi_{\mathbf{P}\mathbf{s}}^{ren}(T)] + C^-[\pi_{-\mathbf{P}\mathbf{s}}^{ren}(T)], \end{aligned} \quad (116)$$

where $\xi_{\mathbf{P}\mathbf{s}}(T)$ and $\pi_{\mathbf{P}\mathbf{s}}^{ren}(T)$ are the following vector distributions:

$$\begin{aligned} (\xi_{\mathbf{P}\mathbf{s}}(T))_{\mathbf{P}'\mathbf{s}'} &= \delta_{\mathbf{P}\mathbf{P}'}(e^{iM_{\mathbf{P}}T}(2M_{\mathbf{P}})^{-1/2})_{\mathbf{s}\mathbf{s}'}, \\ (\pi_{\mathbf{P}\mathbf{s}}^{ren}(T))_{\mathbf{P}'\mathbf{s}'} &= \delta_{\mathbf{P}\mathbf{P}'}(i\Omega^2 M_{\mathbf{P}}^{-2}(M_{\mathbf{P}}/2)^{1/2}e^{iM_{\mathbf{P}}T})_{\mathbf{s}\mathbf{s}'}. \end{aligned} \quad (117)$$

Formulas (115) mean that

$$\begin{aligned} \xi_{\mathbf{P}\mathbf{s}}(T) &= \xi_{\mathbf{P}\mathbf{s}} \cos(\Omega_{\mathbf{P}\mathbf{s}}T) + \pi_{\mathbf{P}\mathbf{s}}^{ren} \frac{\sin(\Omega_{\mathbf{P}\mathbf{s}}T)}{\Omega_{\mathbf{P}\mathbf{s}}} - \\ &\int_0^T \frac{d\tau}{2\sqrt{\omega_{\mathbf{P}/2-s}\omega_{\mathbf{P}/2+s}\Omega_{\mathbf{P}\mathbf{s}}}} \frac{\partial}{\partial\tau} [\sin(\Omega_{\mathbf{P}\mathbf{s}}(T-\tau))] \frac{\partial}{\partial\tau} r_{\mathbf{P}}(\tau), \\ \pi_{\mathbf{P}\mathbf{s}}^{ren}(T) &= \pi_{\mathbf{P}\mathbf{s}}^{ren} \cos(\Omega_{\mathbf{P}\mathbf{s}}T) - \xi_{\mathbf{P}\mathbf{s}} \Omega_{\mathbf{P}\mathbf{s}} \sin(\Omega_{\mathbf{P}\mathbf{s}}T) \\ &- \int_0^T \frac{d\tau}{2} \sqrt{\frac{\Omega_{\mathbf{P}\mathbf{s}}}{\omega_{\mathbf{P}/2-s}\omega_{\mathbf{P}/2+s}}} \frac{\partial}{\partial\tau} [\cos(\Omega_{\mathbf{P}\mathbf{s}}(T-\tau))] \frac{\partial}{\partial\tau} r_{\mathbf{P}}(\tau) \end{aligned} \quad (118)$$

Integrating relations (118) with the function $\varphi \in \mathcal{S}(\mathbf{R}^{2d})$ and applying the operator $(2M_{\mathbf{P}})^{1/2}$, we transform them to the form

$$\begin{aligned} e^{iMT}\bar{\varphi} &= \cos(\Omega T)\bar{\varphi} + iM^{-1}\Omega \sin(\Omega T)\bar{\varphi} \\ &+ \int_0^T d\tau (e^{iM\tau})(\Omega^{-2} - M^{-2})\cos(\Omega(T-\tau))\Omega^2\bar{\varphi}, \end{aligned} \quad (119)$$

$$\begin{aligned} ie^{iMT}M^{-1}\Omega^2\bar{\varphi} &= iM^{-1}\cos(\Omega T)\bar{\varphi} - \Omega \sin(\Omega T)\bar{\varphi} - \\ &\int_0^T d\tau (e^{iMT})(\Omega^{-2} - M^{-2})\Omega^3 \sin(\Omega(T-\tau))\bar{\varphi}. \end{aligned} \quad (120)$$

Here Ω is the operator of multiplication by $\Omega_{\mathbf{P}\mathbf{s}}$. We have used the definition of the operator M^{-2} .

Relation (120) is a corollary of the relation (119) is sufficient to consider the time derivatives of eq.(119). The simplest way to check eq.(119) is to consider the Laplace transformations of the left-hand side

$$\int_0^\infty e^{iMT}e^{-\omega T}dT = \frac{1}{\omega - iM} \quad (121)$$

and of the right-hand side:

$$\frac{\omega}{\omega^2 + \Omega^2} + iM^{-1}\frac{\Omega^2}{\omega^2 + \Omega^2} + \frac{iM}{\omega - iM}(\Omega^{-2} - M^{-2})\Omega^2\frac{\omega}{\omega^2 + \Omega^2}. \quad (122)$$

Formulas (121) and (122) coincide. Thus, eq.(119) is satisfied at $T > 0$, the check procedure for $T < 0$ is analogous. Proposition is proved.

Proposition 4.12. *Relation (114) is satisfied.*

Proof. One has

$$\begin{aligned} e^{iHT}w_2(\mathbf{k}, t_x; \mathbf{p}, t_y)e^{-iHT} &= \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}} Q_{\mathbf{k}+\mathbf{p}; \frac{\mathbf{k}-\mathbf{p}}{2}}(T) \cos(\omega_{\mathbf{k}}t_x + \omega_{\mathbf{p}}t_y) + \\ &\frac{1}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{p}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}} \Pi_{\mathbf{k}+\mathbf{p}; \frac{\mathbf{p}-\mathbf{k}}{2}}(T) \sin(\omega_{\mathbf{k}}t_x + \omega_{\mathbf{p}}t_y) \end{aligned}$$

$$\begin{aligned}
& - \int_0^{t_y} d\tau \frac{\sin(\omega_{\mathbf{p}}(t_y - \tau))}{2\omega_{\mathbf{k}}\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{k}}(t_x - \tau)} Q_{\mathbf{k}+\mathbf{p}}(\tau + T) \\
w_2(\mathbf{k}, t_x + T; \mathbf{p}, t_y + T) &= \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}} Q_{\mathbf{k}+\mathbf{p}; \frac{\mathbf{k}-\mathbf{p}}{2}} \cos(\omega_{\mathbf{k}}(t_x + T) + \omega_{\mathbf{p}}(t_y + T)) + \\
& \frac{1}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{p}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}} \Pi_{\mathbf{k}+\mathbf{p}; \frac{\mathbf{p}-\mathbf{k}}{2}} \sin(\omega_{\mathbf{k}}(t_x + T) + \omega_{\mathbf{p}}(t_y + T)) \\
& - \int_0^{t_x+T} d\tau \frac{\sin(\omega_{\mathbf{k}}(t_x + T - \tau))}{2\omega_{\mathbf{k}}\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}}(\tau - t_y - T)} Q_{\mathbf{k}+\mathbf{p}}(\tau) \\
& - \int_0^{t_y+T} d\tau \frac{\sin(\omega_{\mathbf{p}}(t_y + T - \tau))}{2\omega_{\mathbf{k}}\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{k}}(t_x + T - \tau)} Q_{\mathbf{k}+\mathbf{p}}(\tau) \tag{123}
\end{aligned}$$

It follows from proposition 4.11 that

$$e^{iHT} w_2(\mathbf{k}, t_x; \mathbf{p}, t_y) e^{-iHT} = w_2(\mathbf{k}, t_x + T; \mathbf{p}, t_y + T).$$

We obtain relation (114). Proposition is proved.

5 Poincare invariance of the theory

The purpose of this section is to check the property of relativistic invariance of the theory which mean that:

- (a) the unitary representation of the Poincare group $(\Lambda, a) \mapsto \tilde{U}_{\Lambda, a}$ is constructed:

$$\tilde{U}_{\Lambda_1, a_1} \tilde{U}_{\Lambda_2, a_2} = \tilde{U}_{(\Lambda_1, a_1)(\Lambda_2, a_2)}; \tag{124}$$

- (b) the k -field operators $\tilde{W}_k(x_1, \dots, x_k)$ are Poincare invariant:

$$\tilde{U}_{\Lambda, a} \tilde{W}_k(x_1, \dots, x_k) \tilde{U}_{\Lambda, a}^{-1} = \tilde{W}_k(\Lambda x_1 + a, \dots, \Lambda x_k + a); \tag{125}$$

- (c) the vacuum state is Poincare-invariant:

$$\tilde{U}_{\Lambda, a} |0\rangle = |0\rangle \tag{126}$$

To simplify the problem, remind that the state space has been decomposed according to eq.(44), while the operators $\tilde{U}_{\Lambda, a}$ are looked for in the form $U_{\Lambda, a} \otimes \check{U}_{\Lambda, a}$. The operators $\check{U}_{\Lambda, a}$ have been already constructed in subsection II.F.

First of all, investigate the property of invariance of the operators $\hat{W}_k(x_1, \dots, x_k)$ of the form (95) (at $k \neq 2$). These operators are of the form $1 \otimes \hat{W}_k(x_1, \dots, x_k)$.

Lemma 5.1. 1. The vacuum state is invariant under action of operators $\check{U}_{\Lambda, a}$.

2. For $k \neq 2$, the operators $\hat{W}_k(x_1, \dots, x_k)$ obey the property

$$\check{U}_{\Lambda, a} \hat{W}_k(x_1, \dots, x_k) \check{U}_{\Lambda, a}^{-1} = \hat{W}_k(\Lambda x_1 + a, \dots, \Lambda x_k + a). \tag{127}$$

Proof. The first property is obvious. Prove the second property. It follows from eq.(96), lemma A.11 and formula (48) imply that

$$\begin{aligned}
\check{U}_{\Lambda, a} \hat{W}_s(x_1, \dots, x_s) \check{U}_{\Lambda, a}^{-1} &= \frac{\sqrt{s!}}{(2\pi)^{sd/2}} \int \frac{d\mathbf{p}_1}{\sqrt{2\omega_{\mathbf{p}_1}}} \dots \frac{d\mathbf{p}_s}{\sqrt{2\omega_{\mathbf{p}_s}}} (A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{+(s)} e^{i(p_1 + \dots + p_s)a} e^{i(\Lambda^{-1}\mathbf{p}_1 \cdot x_1 + \dots + \Lambda^{-1}\mathbf{p}_s \cdot x_s)} \\
& \quad + A_{\mathbf{p}_1 \dots \mathbf{p}_s}^{-(s)} e^{-i(p_1 + \dots + p_s)a} e^{-i(\Lambda^{-1}\mathbf{p}_1 \cdot x_1 + \dots + \Lambda^{-1}\mathbf{p}_s \cdot x_s)}). \tag{128}
\end{aligned}$$

Lemma 5.2. Let $U_{\Lambda,a}$ be unitary operators in \mathcal{F} such that

(a) the group property

$$U_{\Lambda_1,a_1}U_{\Lambda_2,a_2} = U_{(\Lambda_1,a_1)(\Lambda_2,a_2)} \quad (129)$$

is satisfied;

(b) the bifield operator is invariant

$$U_{\Lambda,a}\hat{W}_2(x,y)U_{\Lambda,a}^{-1} = \hat{W}_2(\Lambda x + a, \Lambda y + a); \quad (130)$$

(c) the vacuum state is invariant:

$$U_{\Lambda,a}|0\rangle = |0\rangle. \quad (131)$$

Then the operators $\tilde{U}_{\Lambda,a} = U_{\Lambda,a} \otimes \check{U}_{\Lambda,a}$ obey properties (124) - (126).

This lemma is a direct corollary of lemma 5.1 and formula (86) for the operators $\tilde{W}_k(x_1, \dots, x_k)$.

The remaining problem is to construct operators $U_{\Lambda,a}$ satisfying relations (129) - (131). The possible way may be the following. The operators P^μ and M^{mn} in \mathcal{F} have been already constructed. One should then try to construct the operator M^{0k} , check the commutation relations of the Poincare algebra.

However, the following problems arise in the approach. It is not easy to check the self-adjointness of the operator M^{0k} since it is an unbounded operator. Further, to construct the group representation from the algebra representation, one should check the conditions of the Nelson theorem [17] or investigate the properties of analytic vectors [18].

Therefore, another approach will be used for constructing the operators $U_{\Lambda,a}$. First of all, we will check the invariance of the Wightman function

$$\langle 0|\hat{W}_2(x,y)\hat{W}_2(x',y')|0\rangle = \langle 0|\hat{W}_2(\Lambda x + a, \Lambda y + a)\hat{W}_2(\Lambda x' + a, \Lambda y' + a)|0\rangle \quad (132)$$

Then we will define the operator $U_{\Lambda,a}$ from the property

$$U_{\Lambda,a}\hat{W}_2(x_1,y_1)\dots\hat{W}_2(x_k,y_k)|0\rangle = \hat{W}_2(\Lambda x_1 + a, \Lambda y_1 + a)\dots\hat{W}_2(\Lambda x_k + a, \Lambda y_k + a)|0\rangle. \quad (133)$$

This definition will be shown to be correct if and only if the property (132) is satisfied. Let us investigate the properties of the Wightman functions.

5.1 The QQ -propagator

First of all, investigate the vacuum average value $\langle 0|Q(x)Q(y)|0\rangle$. It has the form

$$\langle 0|Q(x)Q(y)|0\rangle = \frac{1}{(2\pi)^{2d}} \int d\mathbf{P} (\lambda\Omega^2\chi, e^{i\mathbf{P}(\mathbf{x}-\mathbf{y})-iM(x^0-y^0)} M^{-1}\lambda\Omega^2\chi). \quad (134)$$

The vector $\lambda\Omega^2\chi$ is viewed as $\lambda_R M^2\chi$. To check the Poincare invariance of the average (134), present it as

$$\langle 0|Q(x)Q(y)|0\rangle = \frac{1}{(2\pi)^{d+1}} \int dP V(P) e^{-iP(x-y)}$$

with

$$V(P^0, \mathbf{P}) = \frac{1}{(2\pi)^{d-1}} \int (\lambda\Omega^2\chi)_{\mathbf{P}\mathbf{s}} (\delta(M_{\mathbf{P}} - P^0) M_{\mathbf{P}}^{-1} \lambda\Omega^2\chi)_{\mathbf{P}\mathbf{s}} d\mathbf{s} \quad (135)$$

Making use of the relations

$$\delta(M_{\mathbf{P}} - P^0) M_{\mathbf{P}}^{-1} = 2\theta(P^0) \delta(M_{\mathbf{P}}^2 - (P^0)^2), \quad 2\pi i \delta(x) = \frac{1}{x - i0} - \frac{1}{x + i0}$$

$$\int d\mathbf{s}(\lambda\Omega^2\chi)_{\mathbf{P}\mathbf{s}} \left(\frac{1}{M_{\mathbf{P}}^2 + \varepsilon^2} \lambda\Omega^2\chi \right)_{\mathbf{P}\mathbf{s}} = \lambda(2\pi)^d \left[1 - \frac{1}{1 + \lambda I(\mathbf{P}, \varepsilon)} \right]$$

we take the formula (135) to the form

$$V(P^0, \mathbf{P}) = 2i\theta(P^0) \left[\frac{\lambda}{1 + \lambda I(\mathbf{P}, i(P^0 + i0))} - \frac{\lambda}{1 + \lambda I(\mathbf{P}, i(P^0 - i0))} \right] \quad (136)$$

where I is of the form (64). We see that the function $\langle 0|Q(x)Q(y)|0 \rangle$ is Poincare invariant.

It will be also necessary to calculate the propagator of the field Q . Formally, one has

$$\langle 0|TQ(x)Q(y)|0 \rangle = \theta(x^0 - y^0) \langle 0|Q(x)Q(y)|0 \rangle - \theta(y^0 - x^0) \langle 0|Q(y)Q(x)|0 \rangle. \quad (137)$$

Eq.(134) implies

$$\langle 0|TQ(x)Q(y)|0 \rangle = \frac{1}{(2\pi)^{2d}} \int d\mathbf{P}(\lambda\Omega^2\chi, e^{i\mathbf{P}(\mathbf{x}-\mathbf{y})-iM|x^0-y^0|} M^{-1} \lambda\Omega^2\chi).$$

The Fourier transformation of the propagator which is defined from the relation

$$\langle 0|TQ(x)Q(y)|0 \rangle = \frac{1}{(2\pi)^{d+1}} \int dPG_Q(P)e^{-iPx}$$

can be presented as

$$G_Q(P^0, \mathbf{P}) = -2i \int d\mathbf{s}(\lambda\Omega^2\chi)_{\mathbf{P}\mathbf{s}} ((M_{\mathbf{P}}^2 - P_0^2 - i0)^{-1} \lambda\Omega^2\chi)_{\mathbf{P}\mathbf{s}} = -2i\lambda + \frac{2i\lambda}{1 + \lambda I(\mathbf{P}, i(P^0 + i0))}$$

We see that formally calculated propagator consists of the singular part

$$\langle 0|TQ(x)Q(y)|0 \rangle^{sing} = -2i\lambda\delta(x - y)$$

and of the regular (renormalized) part with the Fourier transformation

$$G_Q^{ren}(P^0, \mathbf{P}) = \frac{2i\lambda}{1 + \lambda I(\mathbf{P}, i(P^0 + i0))} \quad (138)$$

However, this difficulty is usual: the T -product is defined up to a quasilocal quantity being proportional to $\delta(x - y)$. Note also that the result (138) is in agreement with the approach based on the summation of Feynman graphs [1].

Thus, we have obtained the following result.

Proposition 5.3. *The average value $\langle 0|Q(x)Q(y)|0 \rangle$ is Poincare invariant. Its Fourier transformation has the form (136).*

5.2 The W_2Q -average

The purpose of this subsection is to compute the average values

$$F_1(x, y, z) = \langle 0|\hat{W}_2(x, y)Q(z)|0 \rangle, \quad F_2(x, y, z) = \langle 0|Q(z)\hat{W}_2(x, y)|0 \rangle. \quad (139)$$

However, explicit formulas for the operators \hat{W}_2 are complicated, so that direct calculations are too difficult. Therefore, the indirect method will be used. First of all, these averages will be calculated at $x^0 = y^0 > z^0$ and $z^0 > x^0 = y^0$ correspondingly. Then we will investigate the properties of the Fourier transformation of the averages. The equations on the averages will be obtained. Then the solution of the equations will be found.

Consider the average value $\langle 0|Tw_2(\mathbf{p}, 0; \mathbf{k}, 0)Q_{\mathbf{P}}(t)|0 \rangle$. According to subsection IV,

$$w_2(\mathbf{p}, 0; \mathbf{k}, 0) = \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}} Q_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}}. \quad (140)$$

Therefore,

$$\begin{aligned} & \langle 0|Tw_2(\mathbf{p}, 0; \mathbf{k}, 0)Q_{\mathbf{P}}(t)|0 \rangle \\ &= \frac{1}{(2\pi)^{2d}} \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{2\omega_{\mathbf{k}}\omega_{\mathbf{p}}}} \delta_{\mathbf{k}+\mathbf{p}+\mathbf{P}, 0} [\theta(-t)(e^{iMt}M^{-1}\lambda\Omega^2\chi)_{\mathbf{P}, \frac{\mathbf{k}-\mathbf{p}}{2}} + \theta(t)(e^{-iMt}M^{-1}\lambda\Omega^2\chi)_{\mathbf{P}, \frac{\mathbf{k}-\mathbf{p}}{2}}] \end{aligned} \quad (141)$$

Consider the Fourier transformation of the average (140) defined as

$$G(\mathbf{k}, \mathbf{p}, \varepsilon) \delta_{\mathbf{k}+\mathbf{p}, \mathbf{P}} = \int dt e^{-i\varepsilon t} \langle 0|Tw_2(\mathbf{p}, 0; \mathbf{k}, 0)Q_{\mathbf{P}}(t)|0 \rangle. \quad (142)$$

One has

$$G(\mathbf{k}, \mathbf{p}, \varepsilon) = \frac{1}{(2\pi)^d} \sqrt{\frac{2(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}} \frac{1}{i} \left(\frac{1}{M_{\mathbf{k}+\mathbf{p}}^2 - \varepsilon^2 - i0} \lambda\Omega^2\chi \right)_{\mathbf{k}+\mathbf{p}, \frac{\mathbf{k}-\mathbf{p}}{2}}.$$

Making use of the definition of the operator $M_{\mathbf{P}}^2$, we obtain

$$G(\mathbf{k}, \mathbf{p}, \varepsilon) = -\frac{1}{(2\pi)^d} \frac{2(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}} \frac{1}{(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})^2 - \varepsilon^2 - i0} G_Q^{ren}(\varepsilon, \mathbf{k} + \mathbf{p}).$$

Applying the Fourier transformation to eq.(142), we obtain that

$$\langle 0|T\hat{W}_2(x, y)Q(z)|0 \rangle = -i \int d\xi \langle 0|TQ(\xi)Q(z)|0 \rangle^{ren} \langle 0|T\phi(x)\phi(y)|0 \rangle \langle 0|T\phi(y)\phi(\xi)|0 \rangle, \quad (143)$$

provided that $x = (\mathbf{x}, 0)$, $y = (\mathbf{y}, 0)$, $z = (\mathbf{z}, t)$, while $\langle 0|T\phi(x)\phi(y)|0 \rangle$ is the usual propagator of the free scalar field

$$\langle 0|T\phi(x)\phi(y)|0 \rangle = \frac{1}{(2\pi)^d} \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} e^{-i\omega_{\mathbf{k}}|x^0-y^0|}.$$

Formula (143) is valid not only at $x^0 = y^0 = 0$ but also at $x^0 = y^0 \neq 0$ because of translation invariance properties (section IV). Note also that formula (143) is in agreement with the approach based on the summation of Feynman graphs [1].

Thus, the following result is obtained.

Proposition 5.4. *The Green function $\langle 0|T\hat{W}_2(x, y)Q(z)|0 \rangle$ has the form (143), provided that $x^0 = y^0$.*

5.2.2 Properties of the W_2 -field

Consider the state $\hat{W}_2(x, y)|0 \rangle$. Our purpose is to prove the following property.

Lemma 5.5. *The Fourier transformation*

$$\int dy \hat{W}_2(x, y) e^{-ipy} |0 \rangle \quad (144)$$

vanish at $p^0 < 0$.

Proof. Consider the vector

$$\int dt_y e^{i\varepsilon t_y} w_2(\mathbf{k}, t_x; \mathbf{p}, t_y) |0 \rangle \quad (145)$$

$$C^+[\alpha(\mathbf{k}, t_x; \mathbf{p}, \varepsilon)]|0 >$$

with the following vector α :

$$(\alpha(\mathbf{k}, t_x; \mathbf{p}, \varepsilon))_{\mathbf{P}'s'} = \frac{1}{2} e^{-i\omega_{\mathbf{k}} t_x} \delta_{\mathbf{k}+\mathbf{p}, \mathbf{P}'} \left[\sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}} \omega_{\mathbf{p}}}} (2M_{\mathbf{k}+\mathbf{p}})^{-1/2} - \frac{1}{\sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{p}} (\omega_{\mathbf{k}} + \omega_{\mathbf{p}})}} (M_{\mathbf{k}+\mathbf{p}}/2)^{1/2} + \frac{1}{(2\pi)^d} \frac{1}{2\omega_{\mathbf{k}} \omega_{\mathbf{p}}} \left(\frac{1}{\omega_{\mathbf{k}} + \omega_{\mathbf{p}} + M_{\mathbf{k}+\mathbf{p}}} M_{\mathbf{k}+\mathbf{p}}^{-1/2} \lambda \Omega^2 \chi \right)_{P's'} \right] \quad (146)$$

It follows from the definition of the operator $M_{\mathbf{P}}$ that the quantity (146) vanish. Lemma is proved.

Corollary. *The Fourier transformations*

$$\int F_1(x, y, z) e^{ipx} dp, \quad \int dy e^{-ipy} F_2(x, y, z) \quad (147)$$

vanish at $p^0 < 0$.

5.2.3 Equations for average values

Let us obtain equations for vacuum averages (139). It follows from definition (109) that

$$\left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} + \mu^2 \right) \hat{W}_2(x, y) + Q(x) < 0 | \phi(x) \phi(y) | 0 > = 0.$$

$$\left(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\mu} + \mu^2 \right) \hat{W}_2(x, y) + Q(y) < 0 | \phi(x) \phi(y) | 0 > = 0.$$

Here $< 0 | \phi(x) \phi(y) | 0 >$ is the vacuum average for the free scalar field.

Thus, we obtain the following statement.

Proposition 5.6. *The functions (139) obey the following equations:*

$$\begin{aligned} \left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} + \mu^2 \right) F_1(x, y, z) + < 0 | Q(x) Q(z) | 0 > < 0 | \phi(x) \phi(y) | 0 > &= 0. \\ \left(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\mu} + \mu^2 \right) F_1(x, y, z) + < 0 | Q(y) Q(z) | 0 > < 0 | \phi(x) \phi(y) | 0 > &= 0. \\ \left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} + \mu^2 \right) F_2(x, y, z) + < 0 | Q(z) Q(x) | 0 > < 0 | \phi(x) \phi(y) | 0 > &= 0. \\ \left(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\mu} + \mu^2 \right) F_2(x, y, z) + < 0 | Q(z) Q(y) | 0 > < 0 | \phi(x) \phi(y) | 0 > &= 0. \end{aligned} \quad (148)$$

Let us prove the following lemma.

Lemma 5.7. *Let $F_1(x, y, z)$ and $F_2(x, y, x)$ be distributions obeying the following properties.*

(a) *For some distributions Φ_1 and Φ_2*

$$F_1(x, y, z) = \Phi_1(x - z, y - z), \quad F_2(x, y, z) = \Phi_2(x - z, y - z),$$

(b) *The functions F_1 and F_2 obey eqs.(148).*

(c) *The Fourier transformations (147) vanish at $p^0 < 0$.*

(d) *$F_1(x, y, z) = < 0 | T \hat{W}_2(x, y) Q(z) | 0 >$ at $x^0 = y^0 > z^0$;*

Then

$$F_1(x, y, z) = \langle 0 | \hat{W}_2(x, y) Q(z) | 0 \rangle, \quad F_2(x, y, z) = \langle 0 | Q(z) \hat{W}_2(x, y) | 0 \rangle.$$

Proof. Consider the functions

$$\tilde{F}_1(x, y, z) = F_1(x, y, z) - \langle 0 | \hat{W}_2(x, y) Q(z) | 0 \rangle, \quad \tilde{F}_2(x, y, z) = F_2(x, y, z) - \langle 0 | Q(z) \hat{W}_2(x, y) | 0 \rangle.$$

One has $\tilde{F}_{1,2}(x, y, z) = \tilde{\Phi}_{1,2}(x - z, y - z)$. The functions $\tilde{\Phi}_{1,2}$ obey the following properties:

- (a)
$$\left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} + \mu^2 \right) \tilde{\Phi}_{1,2}(x, y) = 0, \quad \left(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\mu} + \mu^2 \right) \tilde{\Phi}_{1,2}(x, y) = 0,$$
- (b) Fourier transformations $\int dx \tilde{\Phi}_1(x, y) e^{ipx}$ and $\int dy \tilde{\Phi}_1(x, y) e^{-ipy}$ vanish if $p^0 < 0$;
- (c) $\tilde{\Phi}_1(x, y) = 0$ at $x^0 = y^0 > 0$,
 $\tilde{\Phi}_2(x, y) = 0$ at $x^0 = y^0 < 0$.

Properties (a) and (b) mean that

$$\begin{aligned} \tilde{\Phi}_1(x, y) &= \int d\mathbf{k} d\mathbf{p} [\alpha_{\mathbf{k}\mathbf{p}}^+ e^{-i\omega_{\mathbf{k}}x^0 + i\omega_{\mathbf{p}}y^0} + \alpha_{\mathbf{k}\mathbf{p}}^- e^{-i\omega_{\mathbf{k}}x^0 - i\omega_{\mathbf{p}}y^0}] e^{-i\mathbf{k}\mathbf{x} - i\mathbf{p}\mathbf{y}}, \\ \tilde{\Phi}_2(x, y) &= \int d\mathbf{k} d\mathbf{p} [\beta_{\mathbf{k}\mathbf{p}}^+ e^{i\omega_{\mathbf{k}}x^0 + i\omega_{\mathbf{p}}y^0} + \beta_{\mathbf{k}\mathbf{p}}^- e^{-i\omega_{\mathbf{k}}x^0 + i\omega_{\mathbf{p}}y^0}] e^{-i\mathbf{k}\mathbf{x} - i\mathbf{p}\mathbf{y}} \end{aligned}$$

for some $\alpha_{\mathbf{k}\mathbf{p}}^\pm, \beta_{\mathbf{k}\mathbf{p}}^\pm$. Property (c) means that

$$\begin{aligned} \alpha_{\mathbf{k}\mathbf{p}}^+ e^{i\omega_{\mathbf{p}}x^0} + \alpha_{\mathbf{k}\mathbf{p}}^- e^{-i\omega_{\mathbf{p}}x^0} &= 0, \quad x^0 > 0; \\ \beta_{\mathbf{k}\mathbf{p}}^+ e^{i\omega_{\mathbf{k}}x^0} + \beta_{\mathbf{k}\mathbf{p}}^- e^{-i\omega_{\mathbf{k}}x^0} &= 0, \quad x^0 < 0. \end{aligned}$$

We obtain that $\alpha_{\mathbf{k}\mathbf{p}}^\pm = 0, \beta_{\mathbf{k}\mathbf{p}}^\pm = 0$. Therefore, $\tilde{\Phi}_1 = 0, \tilde{\Phi}_2 = 0$. Lemma is proved.

5.2.4 Explicit form of the averages

Lemma 5.8. *The average values have the form*

$$\begin{aligned} F_1(x, y, z) &= \langle 0 | \hat{W}_2(x, y) Q(z) | 0 \rangle = \\ \frac{1}{i} \int d\xi [&(\langle 0 | T\phi(x)\phi(\xi) | 0 \rangle - \langle 0 | \phi(x)\phi(\xi) | 0 \rangle) \langle 0 | Q(\xi) Q(z) | 0 \rangle \langle 0 | \phi(\xi)\phi(y) | 0 \rangle + \\ &\langle 0 | T\phi(y)\phi(\xi) | 0 \rangle - \langle 0 | \phi(y)\phi(\xi) | 0 \rangle) \langle 0 | Q(\xi) Q(z) | 0 \rangle \langle 0 | \phi(x)\phi(\xi) | 0 \rangle + \\ &\langle 0 | \phi(x)\phi(\xi) | 0 \rangle \langle 0 | \phi(y)\phi(\xi) | 0 \rangle \langle 0 | TQ(\xi) Q(z) | 0 \rangle^{ren}] \\ F_2(x, y, z) &= \langle 0 | Q(z) \hat{W}_2(x, y) | 0 \rangle = \\ \frac{1}{i} \int d\xi [&(\langle 0 | T\phi(x)\phi(\xi) | 0 \rangle - \langle 0 | \phi(\xi)\phi(x) | 0 \rangle) \langle 0 | Q(z) Q(\xi) | 0 \rangle \langle 0 | \phi(\xi)\phi(y) | 0 \rangle + \\ &(\langle 0 | T\phi(y)\phi(\xi) | 0 \rangle - \langle 0 | \phi(\xi)\phi(y) | 0 \rangle) \langle 0 | Q(z) Q(\xi) | 0 \rangle \langle 0 | \phi(x)\phi(\xi) | 0 \rangle + \\ &\langle 0 | \phi(\xi)\phi(x) | 0 \rangle \langle 0 | \phi(\xi)\phi(y) | 0 \rangle \langle 0 | TQ(z) Q(\xi) | 0 \rangle^{ren}] \end{aligned} \quad (149)$$

Proof. It is sufficient to check the conditions of lemma 5.7. Condition (a) is obvious. Eqs.(148) are corollaries of the property

$$\left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} + \mu^2 \right) \langle 0 | T\phi(x)\phi(y) | 0 \rangle = \frac{1}{i} \delta(x - y). \quad (150)$$

$$\int dx e^{ipx} < 0 | \phi(x) \phi(\xi) | 0 >, \quad \int dx e^{ipx} < 0 | Q(x) Q(\xi) | 0 >$$

vanish if $p^0 < 0$. The same property is also valid for the function

$$\int dx e^{ipx} < 0 | \phi(x) \phi(\xi) | 0 > < 0 | Q(x) Q(\xi) | 0 > .$$

Since the integral operator with the kernel $< 0 | T \phi(x) \phi(\xi) | 0 >$ multiplies the Fourier transformation by $(\mu^2 - p^2 - i0)^{-1}$, the quantity

$$\int dx e^{ipx} F_1(x, y, z)$$

vanishes if $p^0 < 0$. The analogous property for the function F_2 is checked in the same way. Property (c) is checked.

To check property (d), note that under condition $x^0 = y^0 > z^0$ the function F_1 can be presented as

$$\begin{aligned} \frac{1}{i} \left[\int d\xi (< 0 | T \phi(x) \phi(\xi) - \phi(x) \phi(\xi) | 0 > < 0 | T Q(\xi) Q(z) | 0 >^{ren} < 0 | T \phi(\xi) \phi(y) | 0 > + \right. \\ \int d\xi (< 0 | T \phi(y) \phi(\xi) - \phi(y) \phi(\xi) | 0 > < 0 | T Q(\xi) Q(z) | 0 >^{ren} < 0 | T \phi(x) \phi(\xi) | 0 > \\ \left. + \int d\xi < 0 | \phi(x) \phi(\xi) | 0 > < 0 | \phi(y) \phi(\xi) | 0 > < 0 | T Q(\xi) Q(z) | 0 >^{ren} \right] \end{aligned} \quad (151)$$

since the first and second integrands may be nonzero only at $\xi^0 > x^0 = y^0 > z^0$. The obtained expression coincides with (143). Thus, property (d) is checked for the function F_1 . The check for the function F_2 is analogous. Conditions of lemma 5.7 are checked. This implies lemma 5.8.

5.3 The $W_2 W_2$ -averages

The purpose of this subsection is to find explicit forms of the average value

$$F(x, y : x', y') = < 0 | \hat{W}_2(x, y) \hat{W}_2(x', y') | 0 > \quad (152)$$

We consider first the $x^0 = y^0 > x'^0 = y'^0$ case. Then equations on the function F will be obtained. The solution of this equation will be found.

5.3.1 The equal-time case

Consider the Green function

$$< 0 | T w_2(\mathbf{p}, t; \mathbf{k}, t) w_2(\mathbf{p}', \tau; \mathbf{k}', \tau) | 0 > = \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}} \omega_{\mathbf{p}}}} \sqrt{\frac{\omega_{\mathbf{k}'} + \omega_{\mathbf{p}'}}{\omega_{\mathbf{k}'} \omega_{\mathbf{p}'}}} \delta_{\mathbf{k}+\mathbf{p}; \mathbf{k}'+\mathbf{p}'} \left(\frac{1}{2M_{\mathbf{k}+\mathbf{p}}} e^{-iM_{\mathbf{k}+\mathbf{p}}|t-\tau|} \right)_{\frac{\mathbf{k}-\mathbf{p}}{2}; \frac{\mathbf{k}'-\mathbf{p}'}{2}}$$

and its Fourier transformation

$$\delta_{\mathbf{k}+\mathbf{p}; \mathbf{k}'+\mathbf{p}'} G_2(\mathbf{p}, \mathbf{k}, \mathbf{p}', \mathbf{k}', \varepsilon) = \int dt e^{i\varepsilon t} < 0 | T w_2(\mathbf{p}, t; \mathbf{k}, t) w_2(\mathbf{p}', 0; \mathbf{k}', 0) | 0 > \quad (153)$$

One has

$$G_2(\mathbf{p}, \mathbf{k}, \mathbf{p}', \mathbf{k}', \varepsilon) = \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}} \omega_{\mathbf{p}}}} \sqrt{\frac{\omega_{\mathbf{k}'} + \omega_{\mathbf{p}'}}{\omega_{\mathbf{k}'} \omega_{\mathbf{p}'}}} \frac{1}{i} \left(\frac{1}{M_{\mathbf{k}+\mathbf{p}}^2 - \varepsilon^2 - i0} \right)_{\frac{\mathbf{k}-\mathbf{p}}{2}; \frac{\mathbf{k}'-\mathbf{p}'}{2}} .$$

$$G_2(\mathbf{p}, \mathbf{k}, \mathbf{p}', \mathbf{k}', \varepsilon) = \sqrt{\frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{\omega_{\mathbf{k}}\omega_{\mathbf{p}}}} \sqrt{\frac{\omega_{\mathbf{k}'} + \omega_{\mathbf{p}'}}{\omega_{\mathbf{k}'}\omega_{\mathbf{p}'}}} \frac{1}{2i} \frac{1}{(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})^2 - \varepsilon^2 - i0} (\delta_{\frac{\mathbf{k}-\mathbf{p}}{2}, \frac{\mathbf{k}'-\mathbf{p}'}{2}} + \delta_{\frac{\mathbf{k}-\mathbf{p}}{2}, \frac{\mathbf{p}'-\mathbf{k}'}{2}}) \\ + \frac{1}{(2\pi)^d} \frac{\omega_{\mathbf{k}} + \omega_{\mathbf{p}}}{2\omega_{\mathbf{k}}\omega_{\mathbf{p}}} \frac{\omega_{\mathbf{k}'} + \omega_{\mathbf{p}'}}{2\omega_{\mathbf{k}'}\omega_{\mathbf{p}'}} \frac{1}{(\omega_{\mathbf{k}} + \omega_{\mathbf{p}})^2 - \varepsilon^2 - i0} \frac{1}{(\omega_{\mathbf{k}'} + \omega_{\mathbf{p}'})^2 - \varepsilon^2 - i0} G_Q^{ren}(\varepsilon, \mathbf{k} + \mathbf{p}). \quad (154)$$

Applying the Fourier transformation to expression (153), we obtain that

$$\begin{aligned} & \langle 0 | T \hat{W}_2(x, y) \hat{W}_2(x', y') | 0 \rangle = \\ & \langle 0 | T \phi(x) \phi(x') | 0 \rangle \langle 0 | T \phi(y) \phi(y') | 0 \rangle + \langle 0 | T \phi(x) \phi(y') | 0 \rangle \langle 0 | T \phi(y) \phi(x') | 0 \rangle - \\ & \int d\xi d\xi' \langle 0 | T Q(\xi) Q(\xi') | 0 \rangle^{ren} \langle 0 | T \phi(x) \phi(\xi) | 0 \rangle \langle 0 | T \phi(y) \phi(\xi') | 0 \rangle \\ & \langle 0 | T \phi(\xi') \phi(x') | 0 \rangle \langle 0 | T \phi(\xi) \phi(y') | 0 \rangle \end{aligned} \quad (155)$$

provided that $x^0 = y^0 = t$, $x^{0'} = y^{0'} = \tau$. Eq.(155) is in agreement with the approach based on summation of Feynman graphs.

We have obtained the following statement.

Proposition 5.9. *The Green function F has the form (155) at $x^0 = y^0$, $x^{0'} = y^{0'}$.*

5.3.2 Equations for average values

The following statements are analogs of proposition 5.6 and corollary of lemma 5.5.

Proposition 5.10. *The function F (152) obeys the following equations*

$$\begin{aligned} & \left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} + \mu^2 \right) F(x, y, x', y') = - \langle 0 | \phi(x) \phi(y) | 0 \rangle \langle 0 | Q(x) \hat{W}_2(x', y') | 0 \rangle, \\ & \left(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\mu} + \mu^2 \right) F(x, y, x', y') = - \langle 0 | \phi(x) \phi(y) | 0 \rangle \langle 0 | Q(y) \hat{W}_2(x', y') | 0 \rangle, \\ & \left(\frac{\partial}{\partial x'^\mu} \frac{\partial}{\partial x'^\mu} + \mu^2 \right) F(x, y, x', y') = - \langle 0 | \phi(x') \phi(y') | 0 \rangle \langle 0 | \hat{W}_2(x', y') Q(x') | 0 \rangle, \\ & \left(\frac{\partial}{\partial y'^\mu} \frac{\partial}{\partial y'^\mu} + \mu^2 \right) F(x, y, x', y') = - \langle 0 | \phi(x') \phi(y') | 0 \rangle \langle 0 | \hat{W}_2(x', y') Q(y') | 0 \rangle, \end{aligned} \quad (156)$$

Proposition 5.11. *The Fourier transformations*

$$\int dx e^{ipx} F(x, y, x', y'), \quad \int dy' e^{-ipy'} F(x, y, x', y') \quad (157)$$

vanish at $p^0 < 0$.

Lemma 5.12. *Let $F(x, y, x', y')$ be a distribution obeying the following properties:*

- (a) *the function F obey eq.(156);*
- (b) *the Fourier transformations (157) vanish at $p^0 < 0$;*
- (c) *at $x^0 = y^0 > x^{0'} = y^{0'}$*

$$F(x, y; x', y') = \langle 0 | T \hat{W}_2(x, y) \hat{W}_2(x', y') | 0 \rangle .$$

Then

$$F(x, y; x', y') = \langle 0 | \hat{W}_2(x, y) \hat{W}_2(x', y') | 0 \rangle .$$

$$\tilde{F}(x, y; x', y') = F(x, y; x', y') - \langle 0 | \hat{W}_2(x, y) \hat{W}_2(x', y') | 0 \rangle$$

obeying the properties:

(a)

$$\begin{aligned} \left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} + \mu^2 \right) \tilde{F}(x, y, x', y') &= 0, & \left(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\mu} + \mu^2 \right) \tilde{F}(x, y, x', y') &= 0, \\ \left(\frac{\partial}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu'}} + \mu^2 \right) \tilde{F}(x, y, x', y') &= 0, & \left(\frac{\partial}{\partial y^{\mu'}} \frac{\partial}{\partial y^{\mu'}} + \mu^2 \right) \tilde{F}(x, y, x', y') &= 0, \end{aligned} \quad (158)$$

(b) the Fourier transformations

$$\int dx e^{ipx} \tilde{F}(x, y, x', y'), \quad \int dy' e^{-ip'y'} \tilde{F}(x, y, x', y')$$

vanish at $p^0 < 0$.

(c) $\tilde{F}(x, y; x', y') = 0$ at $x^0 = y^0 > x^{0'} = y^{0'}$.

Therefore,

$$\begin{aligned} \tilde{F}(x, y; x', y') &= \int dk d\mathbf{p} d\mathbf{k}' d\mathbf{p}' [\alpha_{\mathbf{k}\mathbf{p}\mathbf{k}'\mathbf{p}'}^{++} e^{i\omega_{\mathbf{k}}x^0 + i\omega_{\mathbf{p}}y^0 + i\omega_{\mathbf{k}'}x^{0'} - i\omega_{\mathbf{p}'}y^{0'}} + \alpha_{\mathbf{k}\mathbf{p}\mathbf{k}'\mathbf{p}'}^{+-} e^{i\omega_{\mathbf{k}}x^0 + i\omega_{\mathbf{p}}y^0 - i\omega_{\mathbf{k}'}x^{0'} - i\omega_{\mathbf{p}'}y^{0'}} \\ &\quad + \alpha_{\mathbf{k}\mathbf{p}\mathbf{k}'\mathbf{p}'}^{-+} e^{i\omega_{\mathbf{k}}x^0 - i\omega_{\mathbf{p}}y^0 + i\omega_{\mathbf{k}'}x^{0'} - i\omega_{\mathbf{p}'}y^{0'}} + \alpha_{\mathbf{k}\mathbf{p}\mathbf{k}'\mathbf{p}'}^{--} e^{i\omega_{\mathbf{k}}x^0 - i\omega_{\mathbf{p}}y^0 - i\omega_{\mathbf{k}'}x^{0'} - i\omega_{\mathbf{p}'}y^{0'}}] e^{-i(\mathbf{k}\mathbf{x} + \mathbf{p}\mathbf{y} + \mathbf{k}'\mathbf{x}' + \mathbf{p}'\mathbf{y}')} \end{aligned} \quad (159)$$

Relation (c) implies that

$$\alpha_{\mathbf{k}\mathbf{p}\mathbf{k}'\mathbf{p}'}^{++} e^{i\omega_{\mathbf{p}}y^0 + i\omega_{\mathbf{k}'}x^{0'}} + \alpha_{\mathbf{k}\mathbf{p}\mathbf{k}'\mathbf{p}'}^{+-} e^{i\omega_{\mathbf{p}}y^0 - i\omega_{\mathbf{k}'}x^{0'}} + \alpha_{\mathbf{k}\mathbf{p}\mathbf{k}'\mathbf{p}'}^{-+} e^{-i\omega_{\mathbf{p}}y^0 + i\omega_{\mathbf{k}'}x^{0'}} + \alpha_{\mathbf{k}\mathbf{p}\mathbf{k}'\mathbf{p}'}^{--} e^{-i\omega_{\mathbf{p}}y^0 - i\omega_{\mathbf{k}'}x^{0'}} = 0$$

at $x^0 = y^0 > x^{0'} = y^{0'}$. Therefore, all α vanish, so that $\tilde{F} = 0$. Lemma 5.12 is proved.

5.3.3 Explicit form of average values

Lemma 5.13. *The function F has the form*

$$\begin{aligned} F(x, y; x', y') &= \langle 0 | \phi(x) \phi(x') | 0 \rangle \langle 0 | \phi(y) \phi(y') | 0 \rangle + \langle 0 | \phi(x) \phi(y') | 0 \rangle \langle 0 | \phi(y) \phi(x') | 0 \rangle \\ &\quad - i \int d\xi \langle 0 | \phi(x) \phi(\xi) | 0 \rangle \langle 0 | \phi(y) \phi(\xi) | 0 \rangle \langle 0 | TQ(\xi) \hat{W}_2(x', y') | 0 \rangle - \\ &\quad i \int d\xi [\langle 0 | T\phi(x) \phi(\xi) - \phi(x) \phi(\xi) | 0 \rangle \langle 0 | \phi(\xi) \phi(y) | 0 \rangle \\ &\quad + \langle 0 | T\phi(y) \phi(\xi) - \phi(y) \phi(\xi) | 0 \rangle \langle 0 | \phi(x) \phi(\xi) | 0 \rangle] \langle 0 | Q(\xi) \hat{W}_2(x', y') | 0 \rangle \\ &\quad + \int d\xi d\xi' \langle 0 | T\phi(x') \phi(\xi') - \phi(x') \phi(\xi') | 0 \rangle \langle 0 | T\phi(\xi') \phi(y') - \phi(\xi') \phi(y') | 0 \rangle \\ &\quad \times \langle 0 | \phi(x) \phi(\xi) | 0 \rangle \langle 0 | \phi(y) \phi(\xi) | 0 \rangle \langle 0 | TQ(\xi) Q(\xi') | 0 \rangle, \end{aligned} \quad (160)$$

where $\langle 0 | TQ(\xi) \hat{W}_2(x', y') | 0 \rangle$ is the function of the form (143).

Since the straightforward check of the conditions of lemma 5.12 is analogous to proof of lemma 5.8, proof of lemma 5.13 is obvious.

Corollary. *The function F is Poincare-invariant:*

$$F(x, y; x', y') = F(\Lambda x + a, \Lambda y + a, \Lambda x' + a, \Lambda y' + a).$$

First of all, note that all Wightman functions are Poincare invariant.

Lemma 5.14. *The following property is satisfied:*

$$\langle 0 | \hat{W}_2(x_1, y_1) \dots \hat{W}_2(x_n, y_n) | 0 \rangle = \langle 0 | \hat{W}_2(\Lambda x_1 + a, \Lambda y_1 + a) \dots \hat{W}_2(\Lambda x_n + a, \Lambda y_n + a) | 0 \rangle .$$

To prove this lemma, it is sufficient to notice that operators $\hat{W}_2(x, y)$ are linear combinations of creation and annihilation operators, so that the Wick theorem is applicable.

Lemma 5.15.

1. *There exists a unique unitary operator $U_{\Lambda, a}$ obeying the properties: $U_{\Lambda, a} | 0 \rangle = | 0 \rangle$,*

$$U_{\Lambda, a} \hat{W}_2(x_1, y_1) \dots \hat{W}_2(x_n, y_n) | 0 \rangle = \hat{W}_2(\Lambda x_1 + a, \Lambda y_1 + a) \dots \hat{W}_2(\Lambda x_n + a, \Lambda y_n + a) | 0 \rangle \quad (161)$$

2. *The group property (129) is satisfied.*

3. *The invariance property (130) is satisfied.*

Proof. Let $W_2[f] = \int dx dy \hat{W}_2(x, y) f(x, y)$,

$$\Phi = c | 0 \rangle + \sum_{n=1}^N W_2[f_{n,1}] \dots W_2[f_{n,i_n}] | 0 \rangle \quad (162)$$

Set

$$U_{\Lambda, a} \Phi = c | 0 \rangle + \sum_{n=1}^N W_2[u_{\Lambda, a} f_{n,1}] \dots W_2[u_{\Lambda, a} f_{n,i_n}] | 0 \rangle$$

with

$$(u_{\Lambda, a} f)(x, y) = f(\Lambda^{-1}(x - a), \Lambda^{-1}(y - a)).$$

It follows from lemma 5.14 that $(U_{\Lambda, a} \Phi, U_{\Lambda, a} \Phi) = (\Phi, \Phi)$. This means that $U_{\Lambda, a} \Phi = 0$, provided that $\Phi = 0$. Thus, the mapping $U_{\Lambda, a} : \mathcal{D} \rightarrow \mathcal{D}$ is defined (here \mathcal{D} is a set of all vectors of the form (161)). This mapping is a linear isometric (and therefore bounded) operator, while \mathcal{D} is a dense subset of \mathcal{F} . Therefore, the operator $U_{\Lambda, a}$ can be uniquely extended to the space \mathcal{F} . Thus, there exists a unique isometric operator $U_{\Lambda, a}$ obeying the property (161).

Check the group property. Consider the operator

$$V = U_{\Lambda_1, a_1} U_{\Lambda_2, a_2} U_{((\Lambda_1, a_1)(\Lambda_2 a_2))^{-1}}.$$

It satisfies the property:

$$V \hat{W}_2(x_1, y_1) \dots \hat{W}_2(x_n, y_n) | 0 \rangle = \hat{W}_2(x_1, y_1) \dots \hat{W}_2(x_n, y_n) | 0 \rangle$$

Thus, $V = 1$. The group property is checked. One analogously proves that $U_{\Lambda, a}^{-1} = U_{(\Lambda, a)^{-1}}$, so that the isometric operator $U_{\Lambda, a}$ is unitary.

One also has

$$U_{\Lambda, a} \hat{W}_2(x, y) U_{\Lambda, a}^{-1} \hat{W}_2(x_1, y_1) \dots \hat{W}_2(x_n, y_n) | 0 \rangle = \hat{W}_2(\Lambda x + a, \Lambda y + a) \hat{W}_2(x_1, y_1) \dots \hat{W}_2(x_n, y_n) | 0 \rangle .$$

Thus, the property (130) is satisfied on the subspace $\mathcal{D} \subset \mathcal{F}$. Lemma 5.15 is proved.

Thus, we have checked the property of Poincare invariance of the theory.

An old problem of axiomatic and constructive field theory is to construct a nontrivial model of relativistic QFT which obey Wightman axioms. The known models successfully constructed [9] in 2 and 3-dimensional space-time do not contain such difficulties as Stueckelberg divergences and infinite renormalization of the wave function.

Large- N conception enables us to construct a wide class of relativistic quantum theory models. On the one hand, these models are trivial since the Hamiltonian is quadratic with respect to creation and annihilation operators. On the other hand, such phenomena as bound states, quasistationary states and scattering processes are successfully prescribed by the quadratic Hamiltonian of the model.

A suitable language to describe the states and observables of the large- N theory in the leading order of $1/N$ -expansion is the notion of third quantization introduced in quantum cosmology [19, 20] in order to describe processes with variable number of universes.

The third-quantized model considered in this paper may be viewed as a large- N limit of the ordinary field theory. However, it can be also interpreted as an independent model of relativistic quantum theory. We have seen that such properties as renormalizability are satisfied in higher dimensions with respect to ordinary field theories (cf.[21]): the model (47) is renormalized at $d+1 \leq 5$, while the $(\varphi^a \varphi^a)^2$ -model is renormalized at $d+1 \leq 4$ only. Thus, usage of third-quantized models leads to new types of renormalizable theories in higher dimensions.

For the simplicity, we have considered the large- N approximation for the $(\varphi^a \varphi^a)^2$ -model only. One can also consider the $\varphi^a \varphi^a \Phi$ -model. For this case, the phenomenon of infinite renormalization of the wave function should be investigated: it happens that indefinite inner product should be introduced in the state space [22]. Investigation of the large- N QED in the third-quantized formulation gives us a good example of renormalizable gauge theory beyond perturbation theory.

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A Some properties of the Fock space

Let \mathcal{H} be a Hilbert space. Denote by $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \dots \otimes \mathcal{H}$ the n -th tensor degree of space \mathcal{H} . Let π be a transposition (π_1, \dots, π_n) , $1 \leq \pi_1 \neq \dots \neq \pi_n \leq n$ of numbers $(1, \dots, n)$. Consider the operator $\hat{\pi}$ in $\mathcal{H}^{\otimes n}$ which is uniquely defined from the relation

$$\hat{\pi}(e_1 \otimes \dots \otimes e_n) = e_{\pi_1} \otimes \dots \otimes e_{\pi_n}, \quad e_1, \dots, e_n \in \mathcal{H}.$$

By Sym we denote the symmetrization operator

$$Sym \Phi_n = \frac{1}{n!} \sum_{\pi} \hat{\pi} \Phi_n, \quad \Phi_n \in \mathcal{H}^{\otimes n}$$

which is a projector. Introduce the notation $\mathcal{H}^{\vee n} = Sym \mathcal{H}^{\otimes n}$ for the symmetrized n -th tensor degree of \mathcal{H} . Denote also $\mathcal{H}^{\vee 0} = \mathbf{C}$.

Lemma A.1. *The set $\{f^{\otimes n} \equiv f \otimes \dots \otimes f | f \in \mathcal{H}\}$ is a total set in $\mathcal{H}^{\vee n}$.*

Proof. Let $\Phi_n \in \mathcal{H}^{\vee n}$, $\Phi_n \perp f \otimes \dots \otimes f$ for all $f \in \mathcal{H}$. It is necessary to prove that $\Phi_n = 0$. For

$$f = \alpha_1 e_1 + \dots + \alpha_n e_n, \quad \alpha_1, \dots, \alpha_n \in \mathbf{C}, \quad e_1, \dots, e_n \in \mathcal{H}$$

one has

$$0 = \sum_{i_1 \dots i_n=1}^n \alpha_{i_1} \dots \alpha_{i_n} (\Phi_n, e_{i_1} \otimes \dots \otimes e_{i_n}).$$

to zero:

$$n!(\Phi_n, \text{Sym}e_1 \otimes \dots \otimes e_n) = 0. \quad (163)$$

Relation (163) is satisfied for all $e_1, \dots, e_n \in \mathcal{H}$.

Let f_1, f_2, \dots be an orthonormal basis in \mathcal{H} . Then $\{f_{i_1} \otimes \dots \otimes f_{i_n}, i_1, \dots, i_n = \overline{1, \infty}\}$ is an orthonormal basis in $\mathcal{H}^{\otimes n}$. The vector Φ_n can be presented as

$$\Phi_n = \sum_{i_1 \dots i_n = 1}^{\infty} \Phi_{i_1 \dots i_n}^n f_{i_1} \otimes \dots \otimes f_{i_n}.$$

Since

$$\text{Sym}\Phi_n = \sum_{i_1 \dots i_n = 1}^{\infty} \frac{1}{n!} \sum_{\pi} \Phi_{i_{\pi_1} \dots i_{\pi_n}}^n f_{i_1} \otimes \dots \otimes f_{i_n},$$

$\Phi_n \in \mathcal{H}^{\vee n}$ if and only if $\Phi_{i_1 \dots i_n}^n$ is symmetric with respect to transpositions of i_1, \dots, i_n .

For symmetric Φ^n one has

$$(\text{Sym}f_{j_1} \otimes \dots \otimes f_{j_n}, \Phi_n) = \Phi_{j_1 \dots j_n}^n.$$

Thus, eq.(163) implies that $\Phi_{i_1 \dots i_n}^n = 0$ and $\Phi_n = 0$. Lemma is proved.

Definition A.1. The space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\vee n}$$

is a Fock space.

Let $f \in \mathcal{H}$. The creation and annihilation operators $A_n^+(f) : \mathcal{H}^{\vee n-1} \rightarrow \mathcal{H}^{\vee n}$, $A_n^-(f) : \mathcal{H}^{\vee n} \rightarrow \mathcal{H}^{\vee n-1}$ are defined from the relations:

$$\begin{aligned} A_n^+(f)e^{\otimes n-1} &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{\otimes k} \otimes f \otimes e^{\otimes (n-k-1)}, \\ A_n^-(f)e^{\otimes n} &= \sqrt{n}(f, e)e^{\otimes n-1}. \end{aligned} \quad (164)$$

Lemma A.2. The definition (164) is correct. $A_n^{\pm}(f)$ are bounded operators and $\|A_n^{\pm}(f)\| \leq \sqrt{n}\|f\|$.

Proof. One has

$$\begin{aligned} \|A_n^+(f) \sum_i e_i^{\otimes n-1}\| &\leq \frac{1}{\sqrt{n}} \left\| \sum_i e_i^{\otimes k} \otimes f \otimes e_i^{\otimes (n-k-1)} \right\| = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \|f\| \left\| \sum_i e_i^{\otimes n-1} \right\| = \sqrt{n}\|f\| \left\| \sum_i e_i^{\otimes n-1} \right\|. \\ \|A_n^-(f) \sum_i e_i^{\otimes n}\|^2 &= \|A_{n+1}^+(f) \sum_i e_i^{\otimes n}\|^2 - \|f\|^2 \left\| \sum_i e_i^{\otimes n} \right\|^2 \leq n\|f\|^2 \left\| \sum_i e_i^{\otimes n} \right\|^2. \end{aligned}$$

Lemma 2 is proved.

Definition A.2. The operators

$$\begin{aligned} A^+(f)(\Phi_0, \Phi_1, \Phi_2, \dots) &= (0, A_1^+(f)\Phi_0, A_2^+(f)\Phi_1, \dots) \\ A^-(f)(\Phi_0, \Phi_1, \dots) &= (A_1^-(f)\Phi_1, A_2^-(f)\Phi_2, \dots) \end{aligned}$$

are called creation and annihilation operators in the Fock space. The finite vectors of the form $(\Phi_0, \dots, \Phi_n, 0, 0, \dots)$ belong to the domains of $A^{\pm}(f)$.

Definition A.3. The vector $|0\rangle = (1, 0, 0, \dots)$ is a vacuum vector.

Lemma A.3. The vector $f = (0, \dots, 0, \text{Sym}f_1 \otimes \dots \otimes f_n, 0, \dots)$ can be presented as

$$f = \frac{1}{\sqrt{n!}} A^+(f_1) \dots A^+(f_n) |0\rangle$$

Lemma A.4. *The following commutation relations take place*

$$[A^\pm(f_1), A^\pm(f_2)] = 0, \quad [A^-(f_1), A^+(f_2)] = (f_1, f_2).$$

The operators $A^+(f)$ and $A^-(f)$ are conjugated.

Definition A.4. *A coherent state $C(f)$ is a vector $\Phi \in \mathcal{F}$ of the form $C(f) = \Phi = (\Phi_0, \Phi_1, \dots, \Phi_n, \dots)$ with $\Phi_n = \frac{1}{\sqrt{n!}} f^{\otimes n}$.*

Lemma A.5. *The following relations take place:*

$$(C(f), C(f)) = \exp(f, f), \quad A^-(f)C(g) = (g, f)C(g).$$

Lemma A.6. *Let $g \in \mathcal{H}$ and $g_n, n = 1, 2, \dots$ be such a sequence of elements of \mathcal{H} that $\|g_n - g\| \rightarrow_{n \rightarrow \infty} 0$. Then $\|C(g_n) - C(g)\| \rightarrow_{n \rightarrow \infty} 0$.*

Proof. Let $\xi_n = g_n - g$. Then

$$\|C(g + \xi_n) - C(g)\|^2 = e^{(g, g)} [e^{(g, \xi_n)} e^{(\xi_n, g)} e^{(\xi_n, \xi_n)} - e^{(\xi_n, g)} - e^{(g, \xi_n)} + 1] \rightarrow_{n \rightarrow \infty} 0.$$

Lemma A.7. *The set $\{C(f) | f \in \mathcal{H}\}$ is a total set in $\mathcal{F}(\mathcal{H})$.*

Proof. Let $\Phi = (\Phi_0, \Phi_1, \dots, \Phi_n, \dots) \perp C(\alpha f)$ for all $\alpha \in \mathbf{C}$ and $f \in \mathcal{H}$. Therefore,

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} (f^{\otimes n}, \Phi_n) = 0.$$

The series absolutely converges for all α because of the Cauchy-Bunyakovski inequality. Therefore, $(f^{\otimes n}, \Phi_n) = 0$ for all f , so that $\Phi_n = 0$. Lemma is proved.

Lemma A.8. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then there exists a unique isomorphism $I : \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2) \rightarrow \mathcal{F}(\mathcal{H})$ such that*

$$I(C(f_1) \otimes C(f_2)) = C(f_1 \oplus f_2), \quad f_1 \in \mathcal{H}_1, f_2 \in \mathcal{H}_2. \quad (165)$$

Proof. The mapping (165) conserves the inner product. Therefore, formula (165) uniquely defines an isometric operator. Lemma A.7 applied for the space $\mathcal{F}(\mathcal{H})$ implies that the set $I(C(f_1) \otimes C(f_2))$ is a total set in $\mathcal{F}(\mathcal{H})$, so that $I(\mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2)) = \mathcal{F}(\mathcal{H})$. Thus, I is an isomorphism. Lemma is proved.

Lemma A.9. *Let $f = f_1 + f_2$, $f_1 \in \mathcal{H}_1$, $f_2 \in \mathcal{H}_2$. Then*

$$I^{-1}A^\pm(f)I = A^\pm(f_1) \otimes 1 + 1 \otimes A^\pm(f_2). \quad (166)$$

Proof. It is sufficient to note that the matrix elements of the left-hand and right-hand sides of eq.(166) between $C(f_1) \otimes C(f_2)$ and $C(\tilde{f}_1) \otimes C(\tilde{f}_2)$ coincide.

Let U be a bounded operator in \mathcal{H} . By $\mathcal{U}(U)$ we denote the operator $\mathcal{U}(U) : \mathcal{F}(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H})$ of the form

$$\mathcal{U}(U)(f_0, f_1, \dots, f_n, \dots) = (f_0, Uf_1, U \otimes Uf_2, \dots, U^{\otimes n}f_n, \dots).$$

Let \mathcal{H} be a self-adjoint operator in \mathcal{H} . Consider the one-parametric group of unitary operators e^{-iHt} . The operator-valued mapping $t \mapsto \mathcal{U}(e^{-iHt})$ can be also viewed as a one-parametric group. According to the Stone theorem, it has the form

$$\mathcal{U}(e^{-iHt}) = e^{-i\mathcal{F}(H)t}$$

for some self-adjoint operator $\mathcal{F}(H)$ in $\mathcal{F}(\mathcal{H})$. The explicit form of this operator is

$$(\mathcal{F}(H)f)_n = \sum_{k=0}^{n-1} 1^{\otimes k} \otimes H \otimes 1^{\otimes (n-k-1)} f_n.$$

$$Hf = \sum_{ij=1}^{\infty} H_{ij} \varphi_i(\varphi_j, f).$$

Proposition A.10.

$$\mathcal{F}(H) = \sum_{ij=1}^{\infty} H_{ij} A^+[\varphi_i] A^-[\varphi_j].$$

The proof is straightforward.

Lemma A.11. *Let U be an unitary operator in \mathcal{H} , $f \in \mathcal{H}$. Then*

$$\mathcal{U}(U) A^\pm(f) \mathcal{U}(U)^{-1} = A^\pm(Uf). \quad (167)$$

To prove the lemma, it is sufficient to consider the matrix elements of the sides of eq.(167) between coherent states.

Formulate now some results concerning vector and operator distributions. Let $\mathcal{S}(\mathbf{R}^n)$ be a space of complex smooth functions $u : \mathbf{R}^n \rightarrow \mathbf{C}$ such that

$$||u||_{l,m} = \max_{\alpha_1 + \dots + \alpha_n \leq l} \sup_{x \in \mathbf{R}^n} (1 + |x|)^m \left| \frac{\partial^{\alpha_1 + \dots + \alpha_n} u(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right| \rightarrow_{k \rightarrow \infty} 0.$$

We say that the sequence $\{u_k\} \in \mathcal{S}(\mathbf{R}^n)$, $k = \overline{1, \infty}$ tends to zero if

$$||u_k||_{l,m} \rightarrow_{k \rightarrow \infty} 0$$

for all l, m .

Definition A.5. *Let \mathcal{H} be a Hilbert space. A vector distribution f on \mathbf{R}^n is a linear mapping $f : \mathcal{S}(\mathbf{R}^n) \rightarrow \mathcal{H}$ such that $||f(u_k)|| \rightarrow_{k \rightarrow \infty} 0$ if $u_k \rightarrow_{k \rightarrow \infty} 0$.*

Remark. We will write $f(\varphi) = \int dx f(x) \varphi(x)$ and say that $f(x)$ is a vector distribution of the argument $x \in \mathbf{R}^n$.

Lemma A.12. *Let $f : \mathbf{R}^n \rightarrow \mathcal{H}$ be a strongly continuous bounded vector function. Then $f(\varphi) = \int dx f(x) \varphi(x)$ is a vector distribution.*

Lemma A.13. *Let f be a vector distribution. Then $\frac{\partial f}{\partial x^\alpha}$ is a vector distribution.*

The proof is straightforward.

Definition A.6. *Let $\mathcal{D} \subset \mathcal{H}$ be a dense subset of \mathcal{H} . An operator distribution A is a linear mapping*

$$\varphi \in \mathcal{S}(\mathbf{R}^n) \mapsto A(\varphi) : \mathcal{D} \rightarrow \mathcal{D}$$

such that for all $\Phi \in \mathcal{D}$ the mapping $\varphi \mapsto A(\varphi)\Phi$ is a vector distribution.

Let \mathcal{D} be a subset of the Fock space $\mathcal{F}(\mathcal{H})$ which consists of all finite vectors $(f_0, f_1, \dots, f_k, 0, \dots)$.

Lemma A.14. *Let f be a vector distribution. Then $A^\pm(f)$ is an operator distribution.*

Investigate now the cyclic property of the vacuum vector.

Let $\mathcal{G} \in \mathcal{H} \oplus \mathcal{H}$. Consider the operators

$$B(f, g) = A^+(f) + A^-(g), \quad (f, g) \in \mathcal{G}.$$

By $I_1 : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}$ we denote the operator $I_1(f, g) = f$.

Lemma A.15. *Let $I_1 \mathcal{G}$ be a dense subset of \mathcal{H} . Then the set of all linear combinations*

$$\sum_n c_n B(f_{n,1}, g_{n,1}) \dots B(f_{n,k_n}, g_{n,k_n}) |0\rangle, \quad (f_{i,k_i}, g_{i,k_i}) \in \mathcal{G} \quad (168)$$

Proof. Let $\Phi \in \mathcal{F}(\mathcal{H})$. One should prove that it can be approximated by the linear combination (168). Lemma A.7 implies that it is sufficient to prove this statement for the coherent states $C(\varphi)$. Choose such a sequence $f_n \in I_1\mathcal{G}$ that $f_n \rightarrow \varphi$. Lemma A.6 implies that $C(\varphi)$ can be approximated by $C(f_n)$. Furthermore, the coherent state $C(f_n)$ can be approximated a by finite linear combination of vectors $(A^+(f_n))^m|0\rangle$. For some g_n one has $(f_n, g_n) \in \mathcal{G}$. The vector $(A^+(f_n))^m|0\rangle$ can be presented as a linear combination of vectors $(B(f_n, g_n))^k|0\rangle$, Lemma A.15 is proved.

B What is field?

In section IV we have investigated the commutation rule between operators $\frac{1}{N} \sum_{a=1}^N \varphi^a(x_1) \dots \varphi^a(x_k)$ and multifield canonical operator

$$\frac{1}{N} \sum_{a=1}^N \varphi^a(x_1) \dots \varphi^a(x_k) K_N = K_N[(\Phi_0, \phi(x_1) \dots \phi(x_k) \Phi_0) + N^{-1/2} \tilde{W}_k(x_1, \dots, x_k) + O(N^{-1})]$$

The operators $\tilde{W}_k(x_1, \dots, x_k)$ acting in the space (43) of the theory of infinite number of fields were interpreted as multifield operators.

The purpose of this appendix is to construct analogs of the operators $\varphi^a(x)$ in the $N = \infty$ -theory.

One can notice that conception of symmetric states only is not valid for this purpose. If the state $\Psi_N[\varphi^1, \dots, \varphi^N]$ were symmetric with respect to transpositions of the fields $\varphi^1, \dots, \varphi^N$, the state $\varphi^1(\mathbf{x})\Psi_N[\varphi^1, \dots, \varphi^N]$ is not symmetric. Thus, it is necessary to consider the nonsymmetric solutions of eq.(1).

Consider the states of large- N theory which are symmetric with respect to $N - s$ fields $\varphi^{s+1}, \dots, \varphi^N$ only, where s is a finite quantity. Analogously to eq.(6), let Ψ_N be of the form

$$(K_N^s f)[\varphi^1, \dots, \varphi^N] = \sum_{k=0}^{N-s} \frac{\sqrt{k!}}{N^{k/2}} \sum_{s+1 \leq a_1 < \dots < a_k \leq N} f_k[\varphi^1, \dots, \varphi^s, \varphi^{a_1}, \dots, \varphi^{a_k}] \prod_{a > s, a \neq a_1 \dots a_k} \Phi_0[\varphi^a], \quad (169)$$

where $f_k[\varphi^1, \dots, \varphi^s, \phi^1, \dots, \phi^k]$ are functionals being symmetric under transpositions of fields ϕ^1, \dots, ϕ^k and obeying the condition

$$\int D\phi_1 \Phi_0^*[\phi_1] f_k[\varphi^1, \dots, \varphi^s, \phi^1, \dots, \phi^k] = 0. \quad (170)$$

We see that states under consideration are specified by infinite sets

$$\begin{pmatrix} f_0[\varphi^1, \dots, \varphi^s] \\ f_1[\varphi^1, \dots, \varphi^s, \phi^1] \\ \dots \\ f_k[\varphi^1, \dots, \varphi^s, \phi^1, \dots, \phi^k] \\ \dots \end{pmatrix} \quad (171)$$

The state space is then isomorphic to

$$\tilde{\mathcal{F}}_s = \mathcal{H}^{\otimes s} \otimes \mathcal{F}(\oplus_{n=1}^{\infty} \mathcal{H}^{\vee n}). \quad (172)$$

Since the symmetric state can be viewed as a state of the form (169), there should be exist an operator $I_s : \tilde{\mathcal{F}}_0 \rightarrow \tilde{\mathcal{F}}_s$ such that

$$K_N^s I_s f = K_N f.$$

$$\begin{aligned}
(K_N f)[\varphi^1, \dots, \varphi^N] &= \sum_{k=0}^N \frac{\sqrt{k!}}{N^{k/2}} \sum_{1 \leq a_1 < \dots < a_k \leq N} f_k[\varphi^{a_1}, \dots, \varphi^{a_k}] \prod_{a \neq a_1 \dots a_k} \Phi_0[\varphi^a] = \\
&\sum_{k=0}^N \frac{\sqrt{k!}}{N^{k/2}} \sum_{2 \leq a_2 < \dots < a_k \leq N} f_k[\varphi^1, \varphi^{a_2}, \dots, \varphi^{a_k}] \prod_{a \neq 1, a_2, \dots, a_k} \Phi_0[\varphi^a] + \\
&\sum_{k=0}^N \frac{\sqrt{k!}}{N^{k/2}} \Phi_0[\varphi^1] \sum_{2 \leq a_1 < \dots < a_k \leq N} f_k[\varphi^{a_1}, \dots, \varphi^{a_k}] \prod_{a \neq 1, a_2, \dots, a_k} \Phi_0[\varphi^a]. \tag{173}
\end{aligned}$$

We see that

$$(I_1 f)_k[\varphi^1, \phi^1, \dots, \phi^k] = \Phi_0[\varphi^1] f_k[\phi^1, \dots, \phi^k] + N^{-1/2} (\tilde{A}^-[\varphi^1] f)_k[\phi^1, \dots, \phi^k]. \tag{174}$$

One can also perform the symmetrization procedure for the vector (169) and obtain the symmetric state. Therefore, there should exist an operator $S_s : \tilde{\mathcal{F}}_s \rightarrow \tilde{\mathcal{F}}_0$ such that

$$Sym K_N^s f = K_N S_s f.$$

Construct the operator S_1 . If

$$\int D\varphi^1 f_k[\varphi^1, \phi^1, \dots, \phi^k] \Phi_0^*[\varphi^1] = 0, \tag{175}$$

one obtains from direct calculation that

$$(S_1 f)_k[\phi^1, \dots, \phi^k] = N^{-1/2} \int D\varphi (\tilde{A}^+[\varphi] f)_k[\varphi, \phi^1, \dots, \phi^k].$$

If

$$f_k[\varphi^1, \phi^1, \dots, \phi^k] = \Phi_0[\varphi^1] g_k[\phi^1, \dots, \phi^k], \tag{176}$$

then

$$(S_1 f)_k[\phi^1, \dots, \phi^k] = \frac{N-k}{N} g_k[\phi^1, \dots, \phi^k].$$

Generally, f_k can be viewed as a superposition of vectors obeying conditions (175) and (176) correspondingly, so that

$$(S_1 f)_k[\phi^1, \dots, \phi^k] = [1 - \frac{\hat{n}}{N}] \int D\varphi \Phi_0^*[\varphi] f_k[\varphi, \phi^1, \dots, \phi^k] + N^{-1/2} \int D\varphi (\tilde{A}^+[\varphi] f)_k[\varphi, \phi^1, \dots, \phi^k]. \tag{177}$$

The Schrodinger field $\varphi^1(\mathbf{x})$ in $\tilde{\mathcal{F}}_1$ may be viewed as an operator of multiplication by $\varphi^1(\mathbf{x})$, since

$$\varphi^1(\mathbf{x}) K_N^1 = K_N^1 \varphi^1(\mathbf{x}).$$

The multifield can be constructed from the field φ^1 as

$$\tilde{\mathcal{W}}_{N,k}(x_1, \dots, x_k) = S_1 \varphi^1(x_1) \dots \varphi^1(x_k) I_1 \tag{178}$$

since

$$\frac{1}{N} \sum_{a=1}^N \varphi^a(x_1) \dots \varphi^a(x_k) = Sym \varphi^1(x_1) \dots \varphi^1(x_k).$$

One can notice from eqs.(174) and (177) that formula (178) is in agreement with the results of section IV.

To construct the Heisenberg field operator $\varphi^1(x) : \tilde{\mathcal{F}}_1 \rightarrow \tilde{\mathcal{F}}_1$, it is necessary to commute the Hamiltonian operator \mathcal{H}_N with the operator K_N^1 . We obtain:

$$\mathcal{H}_N K_N^1 = K_N^1 \tilde{\mathcal{H}}_N^1, \tag{179}$$

$$\mathcal{H}_N^1 = \mathcal{H}_N + \int d\mathbf{x} \left[-\frac{1}{2} \frac{0}{\delta\varphi^1(\mathbf{x})\delta\varphi^1(\mathbf{x})} + \frac{1}{2} (\nabla\varphi^1)^2(\mathbf{x}) + \frac{\mu}{2} (\varphi^1(\mathbf{x}))^2 \right] + O(N^{-1/2}).$$

Therefore, the Heisenberg operator

$$\varphi^1(\mathbf{x}, t) = e^{i\tilde{\mathcal{H}}_N^1 t} \varphi^1(\mathbf{x}) e^{-i\tilde{\mathcal{H}}_N^1 t}$$

coincides with the operator of the free scalar field up to $O(N^{-1/2})$. Therefore, for operator $\tilde{\mathcal{W}}_{N,k}$ one has

$$\tilde{\mathcal{W}}_{N,k}(x_1, \dots, x_k) = \int D\varphi^1 \Phi_0^*[\varphi^1] \varphi^1(x_1) \dots \varphi^1(x_k) \Phi_0[\varphi^1] + O(N^{-1/2}). \quad (180)$$

This result confirms the hypothesis of section IV.

In order to obtain the explicit form of the $O(N^{-1/2})$ -term of formula (180), it is necessary to compute the $O(N^{-1/2})$ -term in eq.(179). The result is in agreement with section IV.

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